Approved as a Text-book for Schools by the Director of Public Instruction, Burma.

THE NEW
MATRICULATION GEOMETRY

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MATRICULATION GEOMETRY

BOOKS I—V

PREPARED IN ACCORDANCE WITH THE NEW SYLLABUS
SANCTIONED BY GOVERNMENT—
MINISTRY OF EDUCATION

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PREFACE

In preparing THE NEW MATRICULATION GEOMETRY I have followed the Syllabus of the Calcutta University as closely as possible, but not at the sacrifice of simplicity. In the proofs of Theorems, I have always kept in view the excellent suggestions of the Mathematical Association, formerly known as the Association for the Improvement of Geometrical Teaching.

The 'two important Loci' have been given before the theorems on Areas inasmuch as they seem to be the natural sequence of the theorems contained in the first book. Before introducing the Loci, and indeed all important theorems, I have given easy preliminary Exercises, both practical and theoretical, bearing on them to make their nature easily intelligible to the student.

It is now thought, and rightly, that the student will not be fully benefited by the study of a science unless he is trained in both the theory and practice of the subject he learns. With this end in view I have given various Exercises, *theoretical* as well as *numerical and graphical*, under almost every theorem, and the student is recommended to make himself thoroughly familiar with them before beginning a new one.

I am glad to take this opportunity to thank many of my friends, who have helped me from time to time with their valuable criticisms and suggestions, while the book was in preparation.

Any suggestion for improvement, if kindly communicated, will be most thankfully received.

S. C. B.

PREFACE TO THE PRESENT EDITION

While retaining all the features of its previous editions, which made the book so popular throughout India, the book has now been thoroughly revised adapting it to the New Syllabus of Studies for Secondary Schools in Bengal.

BOOK I

ANGLES AT A POINT—PARALLEL STRAIGHT LINES —TRIANGLES—RECTILINEAL FIGURES

THE NEW

MATRICULATION GEOMETRY

BOOK I

PRELIMINARY NOTIONS AND DEFINITIONS

1. Every object, which we see in nature, has length breadth and thickness. Each of these, *viz.*, *length*, *breadth* and *thickness* is called a **dimension**, and a dimension or a combination of dimensions is called a geometrical **magnitude.**

An object is called a body or a solid body or briefly a solid. Hence a **solid** is that which has length, breadth and thickness. A solid has therefore *three* dimensions *viz.*, length, breadth and thickness.

A book, a brick, a piece of stone, &c. are examples of solids.

We shall presently see that all our ideas of geometrical magnitudes are derived from solids.

2. The **volume** of a solid is the amount of space which it occupies.

3. A **surface** is that which has only length and breadth, but no thickness. A surface has therefore *two* dimensions, *viz.*, length and breadth.

A very thin paper may be taken as an example of a surface. But however thin the paper may be, it has still some thickness, and smaller the thickness is, the more nearly do we come to the idea of a surface. Hence the boundaries of a solid are surfaces.

4. A **line** is that which has only length, but no breadth or thickness. It has therefore only *one* dimension *viz.*, length.

If the breadth of a surface be supposed to diminish continually *till it is inappreciable* or nothing, we have a line. Hence,

(i) *the boundaries of a surface are lines* ; and

(ii) *the intersection of two surfaces is a line.*

A geometrical line cannot be actually *drawn* but may be *represented* by the mark made by the pointed end of a pencil moved lightly on paper ; for however thin the mark may be, it has still some breadth and thinner the mark is, the more nearly do we approach to the idea of a line. Hence the trace left by a sharp pencil *is* not a line, but *represents* a line, and *marks* its position on paper.

5. A **point** is that which has position, but no magnitude. A point is therefore said to be of 0 (*i.e.*, no) dimensions.

If the length of a line be supposed to diminish continually till it is nothing, we come to a point. Hence

(i) *the extremities of a line are points* ;
and (ii) *the intersection of two lines is a point.*

A geometrical point cannot be shewn to the eye. It is generally *represented* by the mark made by the sharp end of a pencil pressed lightly on paper ; for however small the mark may be, it has still *some* magnitude. Hence the pencil mark *is* not a point, but *represents* a point, and *marks* its position on paper.

6. Any combination of points, lines or surfaces is called a **figure.**

7. **Geometry** is the science that treats of the properties of figures.

We shall for the present treat of figures which have no more than *two* dimensions.

Lines are of two classes, straight and curved.

8. A **straight line** (or a **right line**) is a line which has the same direction at every point throughout its whole length.

Note. Since a straight line has the *same direction* at every point, there cannot be two straight lines joining the same two points, without coinciding in position ; in other words, *one straight line, and one only*, can be drawn through two given points.

Hence

(i) **Two straight lines cannot enclose a space ;**

(ii) **Two straight lines cannot intersect at more than one point :**

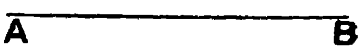
(iii) **Two straight lines cannot have a common segment** or **part.**

and

(iv) **Two points are sufficient to determine a straight line.**

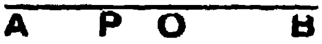
A straight line is usually named by two letters, as **AB**.

When a straight line is drawn between two points **A** and **B**, we are said to join **AB** ; and **AB** denotes the *distance* between **A** and **B**

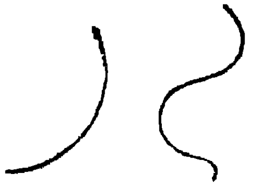


9. A straight line is said to be **bisected** when it is divided into two *equal* parts.

Note. Let a point **P** move from **A** to **B** along **AB**. It is evident that as **P** moves further from **A**, the distance **AP** continually increases and the distance **BP** continually decreases, until **P** comes to a position in which **AP**$=$**BP** ; and there is *only one* such position. Hence *a finite straight line has one, and only one, point of bisection*, as **O** of **AB**.



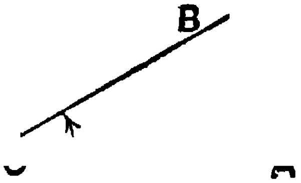
10. A **curved line**, or a **curve**, is a line which continually changes its direction from point to point



11. A **plane surface**, or a **plane**, is a surface in which any two points being taken, the straight line joining them lies wholly in that surface.

Properly speaking this is not a definition of a plane, but rather a *test of its flatness*

12. **Angle**. When two straight lines in a plane meet at a point, they form what is called a **plane angle** or briefly an **angle**.



Thus the two straight lines **AO, BO** meeting at **O** form the angle **AOB.**

The two straight lines which form an angle are called the **arms,** and the point where the arms meet is called the **vertex,** of the angle.

Thus **OA, OB** are the arms and **O** the vertex of the angle **AOB.**

Size of Angle. An angle may be supposed to be formed by the revolution of a line about a point.

Let the line **OB** start from its initial position **OA,** and revolving about **O** in the direction indicated by the arrow, come into the position **OB** ; it forms the angle **AOB.** If it further moves in the same direction, a larger angle will be formed. It is clear therefore that the size of an angle depends on the *amount of turning* necessary to bring the revolving line from the position of one arm of the angle to the position of the other arm ; in other words, an angle is measured by the amount of turning of the revolving arm required to form the angle.

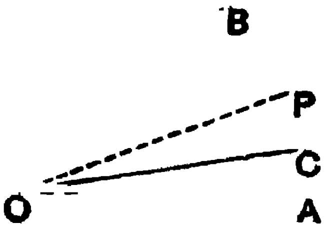
It is to be noticed that the *revolution* of a line does not depend on its length ; hence *the size of an angle does not depend on the length of its arms.*

An angle is generally denoted by three letters of which the letter at the vertex is put in the middle ; as the angle **AOB** or **BOA.** Sometimes when there is no fear of confusion, an angle is denoted simply by the letter at the vertex, as the angle **O** here.

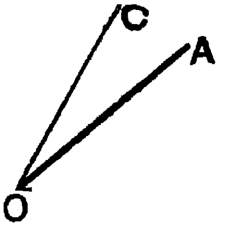
The arms **OA, OB** are said to *contain the angle* **AOB,** and **OB** is said to *make the angle* **AOB** with the straight line **OA.**

13. The **bisector of an angle** is the straight line that divides it into two *equal* angles.

Note. Let **OC** revolve about **O** from the position **OA** to the position **OB.** Then as **OC** revolves, the angle **AOC** continually *increases,* and the angle **COB** continually *diminishes,* until **CO** attains *one* position, and *only one,* in which the angle **AOC** = the angle **COB.** Hence **a given angle has one, and only one, bisector** ; as **OP** is the bisector of the angle **AOB.**



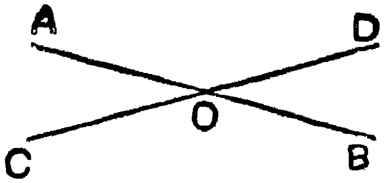
14. Two angles, having the same vertex and lying on either side of a common arm, are called **adjacent angles.**



Thus the angles **AOB**, **AOC**, which have the same vertex **O** and are on opposite sides of the common arm **OA**, are adjacent angles.

15. Of the four angles formed at the point of intersection of two straight lines and of which the point is the *vertex*, any pair of *opposite* angles are called **vertically opposite angles.**

Thus the angles AOC and BOD are vertically opposite angles; so also are the angles AOD and BOC.



16. When one straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and either straight line is called a **perpendicular** to the other.

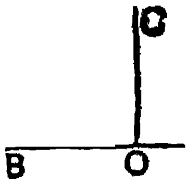


Fig. 1.

Thus the angle **AOC** or the angle **COB**, is a right angle, and **OC** is perpendicular to **AB**, and **AB** is perpendicular to **OC**.

Note. Let **OC** rotate about **O** from the position **OA** into the position **OB**, so that **AOB** is a straight line. It is clear that as **OC** rotates, the angle **AOC** continually *increases* and the angle **COB** continually *decreases*, until **OC** passes through *one* position, and *only one*, in which the angle **AOC** is *equal* to the angle **COB**. Let **OP** be that position, *i.e*, **OC** attains that position only when it is perpendicular to **AB**. Hence.

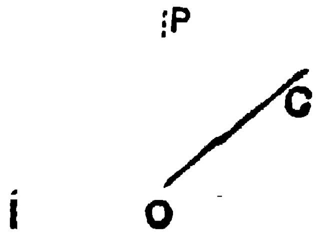


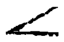
Fig. 2.

(i) All right angles are equal.

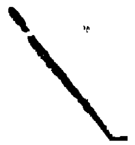
(ii) From a given point in a given straight line one, and only one, perpendicular can be drawn to that line.

A right angle is divided into 90 equal parts called *degrees* (°); a degree into 60 equal parts called *minutes* (′); and a minute into 60 equal parts called *seconds* (″). Thus an angle of 41 degrees 32 minutes and 25 seconds is written $41^\circ\ 32'\ 25''$.

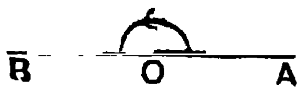
17. An angle which is less than a right *angle is called an* **acute angle.**



18. An angle which is greater than a right angle but less than two right angles is called an **obtuse angle.**



19 When one arm of an angle turns about the vertex until it is in the same *straight line* with the other arm, the angle thus formed is called a **straight angle.**

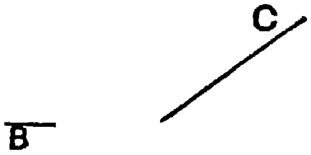


A straight angle is also called a **flat angle.**

Note. 1. It is easy to see that the space occupied by the straight angle **AOB** is the same as that occupied by the two angles **AOC** and **COB** [Fig. 1, Def. 16). Hence

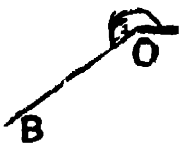
a straight angle=two right angles=180°.

Note 2. Let a straight line rotate about **O**, from the position **OA** into the position **OC** and thence into the position **OB**, so that **AO, OB** are in the same straight line. It has thus described the angles **AOC** and **COB.** But in turning from the position **OA** into the position **OB**, it describes a straight angle, *i.e.*, two right angles. Hence the angles **AOC** and **COB** are together equal to two right angles.

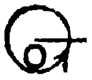


This is a simple proof of Theorem 1.

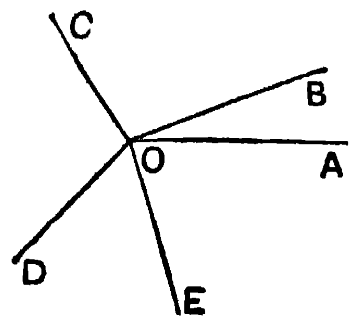
20. An angle which is greater than two right angles but less than four right angles is called a **reflex** or **re-entrant** angle ; as the angle **AOB.**



Note When **OA** makes *one complete revolution* about **O** (*i.e.*, when it starts from its position **OA** and again returns to the same position), it describes **four right angles** or 360° ; as the angle **AOA.**

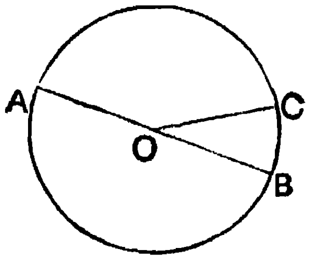


Hence it is clear that if any number of straight lines **OA, OB, OC, OD, OE** meet at **O**, the sum of the angles **AOB, BOC, COD, DOE**, and **EOA** is equal to four right angles. For if **OA**, revolving about **O**, makes one complete revolution, it describes *four* right angles; but in making a complete revolution, **OA** must turn through the angles **AOB, BOC, COD, DOE**, and **EOA**; therefore the sum of these angles is equal to four right angles.



This is a simple proof of Cor. 2, Theorem 3.

21. A **circle** is a plane figure bounded by one line and is such that all straight lines drawn from a certain point within the figure to the bounding line are equal to one another.



The **circumference** of a circle is the line that bounds or encloses the circle.

The **centre** of a circle is that point within it from which all straight lines drawn to the circumference are equal to one another.

Thus the round line is the *circumference*; the surface enclosed by it is the *circle*; and the point **O** is the *centre*.

22. A **radius** of a circle is a straight line drawn from the centre to the circumference, as OA, OB, OC.

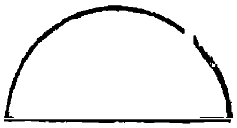
23. A **diameter** of a circle is a straight line drawn through the centre and terminated both ways by the circumference; as AB.

Hence all the radii of a circle are equal; so also are all its diameters. Also a diameter of a circle is *double* its radius.

24. An **arc** of a circle is any portion of the circumference; as AC.

Thus an arc of a circle is a *curved line*.

25. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.



Note that the curved line is a *semi-circumference*, but the surface enclosed by it and the diameter is a *semi-circle*.

POSTULATES

A postulate is an assumption that certain constructions may be performed.

It is taken for granted that

1. *A straight line may be drawn from any one point to any other point.*

2. *A finite* (*i.e.*, terminated) *straight line may be produced to any length in that straight line.*

3. *A circle may be drawn with any point as centre and any length as radius.*

The postulates given above require the use of (i) a *straight ruler*, and (ii) a *pair of compasses*.

By the ruler, constructions mentioned in postulates 1 and 2 are effected, while the compasses enable us to draw a circle of any radius by adjusting them to the length of the radius.

Thus compasses may be used to *transfer distances* from one position to another.

HYPOTHETICAL CONSTRUCTIONS

[These are constructions that are assumed like the Postulates, as being necessary for the demonstration of Theorems.]

Let it be supposed that

1. *A finite straight line may be bisected at a point.*

2. *A straight line may be drawn bisecting a given angle.*

3. *A straight line may be drawn perpendicular to a given straight line from any point in it.*

4. *A parallel to a given straight line may be drawn through a given point.*

5. *A straight line may be drawn making a given angle with a given straight line at a given point in it.*

AXIOMS

An **axiom** is a statement the truth of which is so obvious as to be admitted without proof. In other words, *an axiom is a self-evident truth.*

Axioms are either (i) *general,* i.e., those that are applicable to magnitudes of all kinds, or (ii) *geometrical,* i.e., those that relate to geometrical magnitudes only.

GENERAL AXIOMS

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Things which are the same multiples of equals are equal.

For example : **Doubles of equal things are equal.**

7. Things which are the same parts of equals are equal.

For examples : **Halves of equal things are equal.**

8. The whole is greater than its part.

GEOMETRICAL AXIOMS

9. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal.

The method of placing one magnitude upon another to see whether they coincide or not, is called **superposition**, and the magnitude that is placed upon another is said to be **applied** to the other.

10. Two straight lines cannot enclose a space.

This is inferred from Definition 8.

11. All right angles are equal to one another.

This is easily seen from Definition 16.

PROPOSITIONS

A statement which proposes to prove some geometrical truth, or to effect some geometrical construction, is called a **Proposition.**

A proposition is either a *Theorem* or a *Problem.*

A **Theorem** is one in which some geometrical truth is to be established or proved.

A **Problem** is one in which some geometrical construction is to be performed ; such as to draw some particular line, or to make a required figure.

A proposition usually consists of *four* parts—*viz.*, (i) the *Enunciation*, (ii) the *Particular Enunciation*, (iii) the *Construction*, and (iv) the *Proof.*

In a proposition, these parts follow one another in the order given.

(i) The **Enunciation** is the general statement of what is to be proved or done.

(ii) The **Particular Enunciation** is the repetition of the statement in the enunciation with reference to a figure.

(iii) The **Construction** effects the drawing of such lines and circles as may be necessary for the proof of a theorem or the solution of a problem.

(iv) The **Proof** shews that the theorem is true or that the problem has been solved.

In some theorems, however, no *construction* is required, while in some others, the *construction* and *proof* are combined.

The Enunciation of a *Theorem* has two parts—the **hypothesis** or that which is assumed, and the **conclusion** or that which is to be proved.

The Enunciation of a *Problem* has two parts—the **data** or things that are given, and the **quaesita** or things that are required to be done.

A **Corollary** to a theorem is a statement the truth of which readily follows from the theorem.

The following symbols and abbreviations are used in the text of this book and may be used in writing out the propositions.

SYMBOLS

$\therefore$	*for* therefore.	$\angle$	*for* angle *or* the angle.
$\because$	... since, *or* because.	rt. $\angle$	... right angle.
$=$	... is *or* are equal to, *or* equal to.	$\triangle$	... triangle, *or* the triangle.
$\equiv$	... is congruent with.	$\odot$	... circle.
$+$	... and *or* together with.	$\bigcirc^{ce}$	... circumference.
$>$	... is *or* are greater than.	$\perp$	... perpendicular.
$<$	... is *or* are less than.	$\parallel$	... parallel.
		$\parallel^{m}$	... parallelogram.

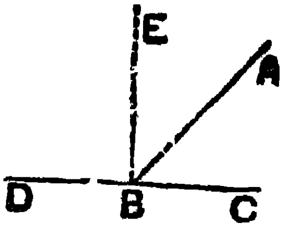
ABBREVIATIONS

adj.	*for* adjacent.	post.	*for* postulate.
alt.	... alternate.	prob.	... problem.
ax.	... axiom.	prop.	... proposition.
const.	... construction.	pt.	... point.
cor.	... corollary.	quadl.	... quadrilateral.
def.	... definition.	rect.	... rectangle.
diag.	... diagonal.	rectil.	... rectilineal.
ext.	... exterior.	reqd.	... required.
fig.	... figure.	sq.	... square.
hyp.	... hypothesis.	st.	... straight.
hypot.	... hypotenuse.	theor.	... theorem.
int.	... interior.	AB^2	... square on AB.
opp.	... opposite.	rect. AB, CD $AB \cdot CD$	... rectangle contained by AB and CD.
parl.	... parallel.		
parm.	... parallelogram.	(A)	... circle of centre A.
perp.	... perpendicular.		

ANGLES AT A POINT

THEOREM 1. [Euc. I. 13.]

If a straight line stands on another straight line, the sum of the two adjacent angles is equal to two right angles.



Let the st. line AB stand on the st. line CD and form the two adjacent $\angle^s$ ABC, ABD.

To prove that

the $\angle$ABC + *the* $\angle$ABD = 2 *right angles.*

Construction. Suppose BE is drawn perpendicular to CD.

Proof. The $\angle$ABC + the $\angle$ABD

= the $\angle$ABC + the $\angle$ABE + the $\angle$EBD;

also the $\angle$CBE + the $\angle$EBD

= the $\angle$ABC + the $\angle$ABE + the $\angle$EBD;

∴ the $\angle$ABC + the $\angle$ABD = the $\angle$CBE + the $\angle$EBD.

But the $\angle^s$ CBE, EBD are right angles. [*Const.*

∴ the $\angle$ABC + the $\angle$ABD = 2 right angles. Q.E.D.*

[For **Proof by Rotation**, see Def. 19, *Note 2.*]

Def. 1. If two angles are together equal to two right angles, each is called the **supplement** of the other; and the two angles are said to be **supplementary** to each other. Thus the $\angle$s **ABC, ABD** are supplementary angles.

Def. 2. If two angles are together equal to one right angle, each is called the **complement** of the other; and the two angles are said to be **complementary** to each other. Thus the $\angle$s **ABC, ABE** are complementary angles.

---

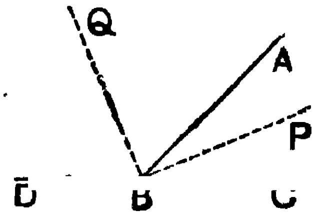
[*] These letters stand for 'Quod erat demonstrandum' (*which was to be proved*).

Cor. 1. *The three angles which any two straight lines* (AB, EB) *make on one side of a third straight line* (CD) *at the same point* (B), *are together equal to two right angles.*

Cor. 2. *The supplements, or the complements, of the same angle or of equal angles, are equal.*

Cor. 3. *The bisectors of the adjacent angles, which one straight line makes with another, are at right angles.*

Let **BP**, **BQ** be the bisectors of the angles **ABC**, **ABD**.



Then the $\angle$ PBQ shall be a rt. $\angle$.
Now the $\angle$ ABP $= \frac{1}{2}$ the $\angle$ ABC,
and the $\angle$ ABQ $= \frac{1}{2}$ the $\angle$ ABD ;
$\therefore$ the $\angle$ ABP + the $\angle$ ABQ, *i.e.*,
the $\angle$ PBQ $= \frac{1}{2}$ (the $\angle$ ABC + the $\angle$ ABD)
$= \frac{1}{2}$ of 2 rt. $\angle$s $=$ one rt. $\angle$.

Def. 3. If **ABC** be an angle whose arm **BC** is produced to **D**, then the lines **BP**, **BQ**, which bisect the adjacent angles **ABC**, **ABD**, are respectively called the **internal** and **external bisectors** of the **angle ABC**.

We may now enunciate the above Corollary thus :—**The internal and external bisectors of an angle are at right angles.**

EXERCISES I.

1. From a point **C** in **AB**, straight lines **CD**, **CE** are drawn on opposite sides of **AB**. If the angles **ACD**, **ACE** are equal, shew that the angles **BCD**, **BCE** are also equal.

2. If two interior angles of a triangle are equal, the corresponding exterior angles are also equal.

3. Two straight lines cannot have a common segment.

4. If one of the four angles made by two intersecting straight lines be a right angle, the others are also right angles.

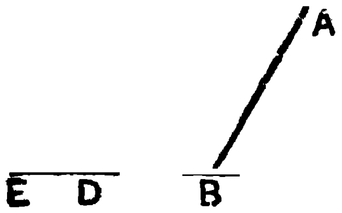
5. In the Fig. of Theor. 1, prove that the angle **ABE** is half the difference of the angles **ABC**, **ABD**.

6. If a st. line **BP** bisects an angle **ABC**, then the st. line **BQ** at right angles to it bisects the adjacent angle **ABD** [see the Fig. above].

THEOREM 2 [Euc. I. 14.]

[Converse of Theorem 1.]

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in the same straight line.



At B in the st. line AB, let the two st. lines BC, BD, on opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.

To prove that BC *and* BD *are in the same straight line.*

Construction. Produce CB to E.

Proof. Because AB stands on the st. line CBE ; [*Const.*

∴ the $\angle$ABC + the $\angle$ABE = 2 right angles. [I. 1.

But the $\angle$ABC + the $\angle$ABD = 2 right angles. [*Hyp.*

∴ the $\angle$ABC + the $\angle$ABE = the $\angle$ABC + the $\angle$ABD.

From these equals take away the $\angle$ABC ;

∴ the $\angle$ABE = the $\angle$ABD ;

∴ BD lies along BE.

But BC and BE are in the same st. line ; [*Const.*

∴ BC and BD are in the same st. line.

Q. E. D.

Note.

Two theorems are said to be **converse** *of one another, when the hypothesis of the one is the conclusion of the other*; and *vice versâ.*

Theorem 2 is the *converse* of Theorem 1, as the student will see from the following scheme.

THEOREM 1.

Hyp.	**Concl.**
One st. line makes with another st. line *two angles.*	*The sum of the two angles is 2 rt. angles.*

THEOREM 2.

Two st. lines make with another st. line *two angles whose sum is two rt. angles.*	*The two st. lines are one st. line.*

The converse of Theorem 1, namely, Theorem 2, is true. It should however be observed that the converse of *every* Theorem is *not necessarily true.*

EXERCISES II.

1. If from a point **O** in a straight line **AB**, **OP** and **OQ** are drawn at right angles to **AB** on opposite sides, shew that **POQ** is a straight line.

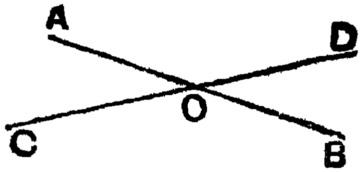
2. If each of the four angles, made by four straight lines which meet at a point, is a right angle, the four straight lines form two straight lines.

3. **If from a point O in a straight line AB, two lines OP, OQ are drawn on opposite sides of AB, so as to make the angles AOP, BOQ equal, prove that POQ is a straight line.**

4. **AOB**, **AOC** are two adjacent angles having a common arm **OA**; if their bisectors are at right angles, shew that **BOC** is a straight line.

THEOREM 3. [Euc. I. 15.]

If two straight lines intersect, the vertically opposite angles are equal.



Let the straight lines AB, CD intersect at O.

To prove that (i) *the* ∠AOC = *the* ∠BOD;

(ii) *the* ∠AOD = *the* ∠BOC.

Proof. Because AO stands on the st. line CD,

∴ the ∠AOC + the ∠AOD = 2 right angles. [I. 1.

And because DO stands on the st. line AB,

∴ the ∠BOD + the ∠AOD = 2 right angles. [I. 1.

∴ the ∠AOC + the ∠AOD = the ∠BOD + the ∠AOD.

From these equals, take away the common ∠AOD;

∴ the ∠AOC = the ∠BOD.

Similarly it can be proved that

the ∠AOD = the ∠BOC. Q. E. D.

Cor. 1. *If two straight lines intersect, the sum of the four angles thus formed is equal to four right angles.*

Cor. 2. *If any number of straight lines meet at a point, the sum of all the angles thus formed is equal to four right angles.* [See Fig. Def. 20, Note.]

EXERCISES III.

1. In the diagram of Theorem 3, if EO bisects the angle AOC, it, when produced, also bisects the angle BOD.

2. **If two straight lines intersect, the bisectors of opposite vertical angles are in the same straight line.**

3. The bisectors of the four angles, which one straight line makes with another, form two straight lines at right angles to one another.

4. **If four straight lines meet at a point so as to make the opposite angles equal, these straight lines are, two and two, in the same straight line.**

EXERCISES IV.

Write down the complement of an angle of

(i) 18°, (ii) 30°, (iii) 45°, (iv) 54°,
(v) 75°, (vi) $40^\circ\ 12'$, (vii) $56^\circ\ 10'\ 35''$, (viii) $83^\circ\ 24'\ 48''$,
(ix) $58'\ 46''$, (x) $51''$.

2. Write down the supplement of an angle of

(i) 25°, (ii) 60°, (iii) 85°, (iv) 108°, (v) 120°,
(vi) 150°, (vii) $120^\circ\ 5'\ 30''$, (viii) $18'\ 50''$, (ix) $25''$.

3. In the diagram of Cor. 3 to Theorem 1, prove that the $\angle$**CBP** is the complement of the $\angle$**DBQ**.

If the $\angle$**ABP** $= 18^\circ$, find the angles **DBQ**, **ABD** and **ABC**.

4. In the diagram of Theorem 1, if the angle **ABD** is double of the angle **ABC**, find the magnitude of each in degrees.

5. Two angles are complementary and one of them is treble of the other; write down their values in degrees.

6. **AOB**, **BOC** are two adjacent angles having the same vertex **O**; if **OP** is the bisector of the angle $\angle$**BOC**, prove that

$$2\angle AOP = \text{the } \angle AOB + \text{the } \angle AOC.$$

7. Four lines **AO**, **BO**, **CO**, **DO** meet at a point; if the angles **AOB**, **COD** are equal and the lines **AO**, **CO** form one straight line, prove that **BO**, **DO** also form one straight line.

If the $\angle$**AOB** $= \frac{1}{4}$ the $\angle$**BOC**, find the magnitude of each angle.

TRIANGLES

DEFINITIONS

1. A **plane figure** is a portion of a plane surface bounded by one or more lines, straight or curved.

2. The **area** of a plane figure is the amount of surface enclosed by its boundaries.

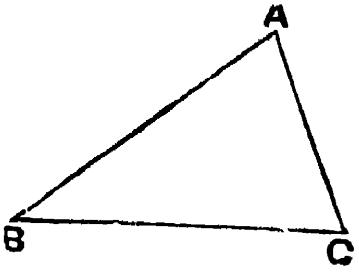
3. A **rectilineal figure** is a plane figure bounded by straight lines (right lines).

The bounding straight lines are called its **sides**, and the **sum** of the sides is called its **perimeter**.

4. A **triangle** is a plane figure bounded by *three* straight lines.

A triangle is named by three letters placed at its angular points; as **ABC**.

Any one of the angular points of a triangle may be called a **vertex**, and the opposite side is then called the **base**. Thus if **A** is a vertex, **BC** is the base; if **B** is a vertex, **AC** is the base; or if **C** is a vertex, **AB** is the base.

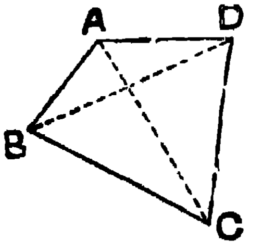


If two sides of a triangle are mentioned, the third side is often called the **base**. Thus if **AB**, **AC** are sides, **BC** is the base; or if **BA**, **BC** are sides **AC** is the base; etc.

5. A **quadrilateral** is a plane figure bounded by *four* straight lines.

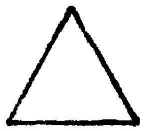
A quadrilateral is named by letters placed at its angular points; as **ABCD**.

A **diagonal** of a quadrilateral is the straight line that joins its opposite angular points. A quadrilateral has thus *two* diagonals; as **AC**, **BD**.

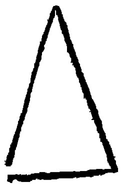


Triangles, when classified with reference to their *sides* are called *equilateral*, *isosceles* and *scalene* triangles.

6. An **equilateral** triangle is one which has three equal sides.

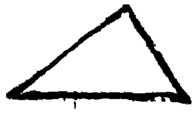


7. An **isosceles** triangle is one which has two equal sides.



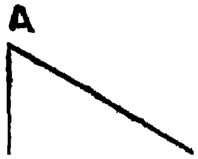
The point of intersection of the equal sides of an isosceles triangle is usually called its *vertex* and the opposite side (*i. e., the side other than the equal sides*) is called its *base*. Thus the *vertical angle* of an isosceles triangle is the angle contained by the equal sides.

8. A **scalene** triangle is one which has three unequal sides.



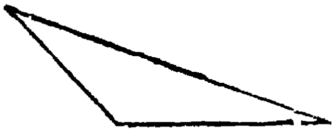
Triangles, when classified with reference to their *angles* are called *right-angled*, *obtuse-angled* and *acute-angled* triangles.

9. A **right-angled** triangle is one which has one right angle.

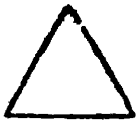


The side opposite to the right angle in a right-angled triangle is called the **hypotenuse** ; as **AC**.

10. An **obtuse-angled** triangle is one which has one obtuse angle.



11. An **acute-angled** triangle is one which has three acute angles.



Note. A triangle has *six* elements or parts, viz., *three* sides and *three* angles ; it has also an *area*.

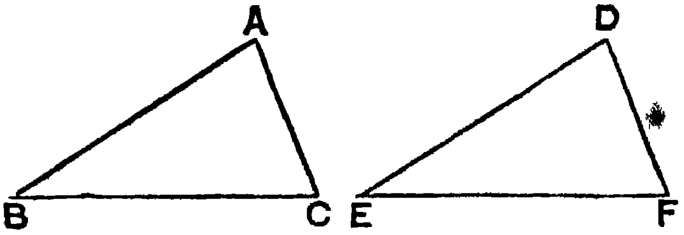
Two triangles **ABC, DEF** [Figs. p. 23.] are **equal in all respects**, if the six parts and area of one are equal to the six parts and area of the other ; [that is, if **AB = DE, BC = EF, CA = FD, ∠A = ∠D, ∠B = ∠E, ∠C = ∠F** and the area of **ABC** = the area of **DEF**. When two triangles are *equal in all respects*, they are said to be **congruent** or **identically equal**.

If the triangles **ABC, DEF** are such that by **superposition** (*i.e.*, placing one upon the other) they may be made to **coincide**, *i.e.*, fit exactly, the triangles are congruent ; for as they *fill the same space*, their areas are obviously equal and the sides and angles of one are equal to the sides and angles of the other.

The symbol ≡ is used to denote the congruence or identical equality of two magnitudes ; thus the **△ABC ≡** the **△DEF** means that the two triangles are congruent.

THEOREM 4. [Euc. I. 4.]

If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, the two triangles are congruent.



Let ABC, DEF be two triangles, in which

(i) AB=DE,

(ii) AC=DF,

and (iii) the included ∠BAC=the included ∠EDF.

To prove that *the* △ˢABC, DEF *are congruent.*

Proof. Apply the △ABC to the △DEF, so that the pt. A falls on the pt. D and the side AB along the side DE.

Then because AB=DE ; [*Hyp.*

∴ the pt. B coincides with the pt. E.

And because AB falls along DE,

and the ∠A=the ∠D ; [*Hyp.*

∴ AC falls along DF.

Then because AC=DF, [*Hyp.*

∴ the pt. C coincides with the pt. F.

Thus B coinciding with E and C with F,

the side BC must coincide with the side EF.

Hence the △ABC coincides with the △DEF ;

that is, the △ˢ ABC, DEF are congruent. Q.E.D.

Note. The △ˢ ABC, DEF are congruent, for they coincide with one another, so that besides the three parts that are given equal, *viz.*, (i) AB=DE, (ii) AC=DF,

(iii) the ∠A = the ∠D, we find that the remaining *three* parts are also equal, *viz.*,—

(1) BC = EF.

[That is, *the third sides are equal*]

(2) The ∠ABC = the ∠DEF.

(3) The ∠ACB = the ∠DFE.

[That is, *the angles opposite to the equal sides are equal.*]

Also the area of the △ABC = the area of the △DEF.

Def. The sides *opposite* to the equal angles in equiangular triangles are called **corresponding** sides; thus **AB** and **DE**, **AC** and **DF**, **BC** and **EF** are pairs of corresponding sides.

EXERCISES V.

1. **The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.**

2. **If the perpendicular from the vertex to the base of a triangle bisects the base, the triangle is isosceles.**

3. The diagonals of a square are equal.

[A *square* is a quadrilateral having all its sides equal and all its angles right angles.]

4. If **CD** be perpendicular to **AB** at its middle point **D**, shew that **AC** = **BC**.

5. **If two st. lines bisect each other at right angles, any point in either is equidistant from the extremities of the other.**

6. **ABC** is a triangle; **DO**, **EO** are drawn bisecting **AB** and **AC** at right angles and meeting in **O**; prove that **OA** = **OB** = **OC**.

7. If two st. lines **AB**, **CD** bisect each other at **O**, shew that (i) the △**AOC** ≡ the △**BOD** and (ii) the △**AOD** ≡ the △**BOC**.

8. In the equal sides **AB**, **AC** of an isosceles triangle, or in those sides produced, are taken points **D** and **E** respectively, such that **AD** = **AE**; prove that **CD** = **BE**.

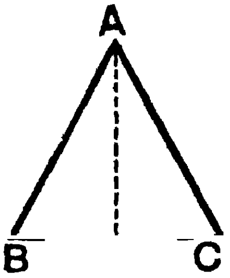
9. **ABCD** is a quadrilateral in which **AB** = **CD** and the ∠**ABC** = the ∠**BCD**; prove that its diagonals are equal.

10. If two adjacent sides of a quadrilateral be equal and the diagonal bisect the angle between them, prove that (i) their other sides are equal, and (ii) this diagonal bisects the other diagonal perpendicularly.

11. If the straight line joining the middle points of two opposite sides of a quadrilateral be perpendicular to each of these sides, prove that its other two sides are equal.

THEOREM 5. [Euc. I. 5.]

If two sides of a triangle are equal, the angles opposite to these sides are equal.



Let ABC be a triangle in which AB=AC.

To prove that *the* ∠ACB=*the* ∠ABC.

Construction. Let AD bisect the ∠BAC; and let it meet BC in D,

Proof. In the Δ^s BAD, CAD,

(i) AB=AC.

(ii) AD is common to both triangles,

and (iii) the included ∠BAD=the included ∠CAD;

∴ the traingles are congruent; [I. 4.

so that the ∠ABD=the ∠ACD,

that is, the ∠ABC=the ∠ACB.

Q. E. D.

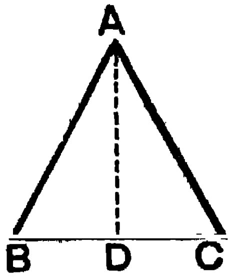
Note. This Theorem may be enunciated otherwise thus: **The angles at the base of an isosceles triangle are equal.**

Cor. 1. *If the equal sides* (AB, AC) *are produced, the angles on the other side of the base* (BC) *are equal.*

Cor. 2. *If a triangle is equilateral, it is also equiangular.*

THEOREM 5. [ALTERNATIVE PROOF]

If two sides of a triangle are equal, the angles opposite to these sides are equal.



Let ABC be a triangle, in which AB = AC.

To prove that *the* ∠ ACB = *the* ∠ ABC.

Construction. Let AD bisect the ∠BAC, and let it meet BC in D.

Proof. Fold the △ABC about the bisector AD.

Then ∵ the ∠BAD = the ∠CAD,
∴ AB falls along AC ;
and ∵ AB = AC,
∴ B coincides with C ;
∴ DB coincides with DC ;
∴ the ∠ABD coincides with the ∠ACD ;
that is, the ∠ABC = the ∠ACB.

Q. E. D.

AXIS OF SYMMETRY

Def. If the parts of a figure, when the figure is folded about a line in it, coincide with one another, that line is called an **axis of symmetry** ; and the figure is said to be *symmetrical about the line.*

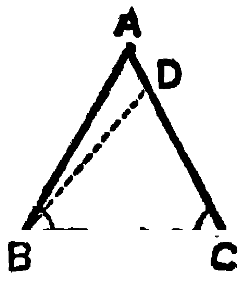
Thus in the above figure, AD is the axis of symmetry of the △ABC. So also a diagonal is an axis of symmetry of a square ; a diameter is of a circle ; and so on.

Hence an isosceles triangle has *one*, a rhombus has *two*, an equilateral triangle has *three*, a square has *four*, and a circle has *any number* of, axes of symmetry.

THEOREM 6. [Euc. I. 6.]

[Converse of Theorem 5.]

It two angles of a triangle are equal, the sides opposite to these angles are equal.



Let ABC be a triangle, in which the ∠ ABC = the ∠ ACB.

To prove that AC = AB.

Construction. If AC is not equal to AB, suppose AC to be the greater.

From AC, cut off CD equal to AB. Join BD.

Proof. In the △^{s} DCB, ABC,

(i) DC = AB,

(ii) BC is common to both,

and (iii) the included ∠ DCB = the included ∠ ABC ;

∴ the △ DCB = the △ ABC, [I. 4.

the part equal to the whole ; which is absurd.

∴ AC is not greater than AB ;

i.e., AC = AB.

Q.E.D.

Cor. 1. *If the exterior angles formed by producing any two sides of a triangle are equal, these two sides are equal.*

Cor. 2. *If a traingle is equiangular, it is also equilateral.*

EXERCISES VI.

1. The triangle, formed by joining the middle points of the sides of an isosceles triangle, is also isosceles.

2. If the base of a triangle be produced both ways and the exterior angles thus formed be equal to one another, then the triangle is isosceles.

3. If a four-sided figure be equilateral, its opposite angles are equal.

4. The straight lines joining the middle point of the base of an isosceles triangle to the middle points of the sides are equal.

5. **P, Q** are points in the base **BC** of an isosceles triangle **ABC**, such that **BP**=**CQ**. Shew that **APQ** is an isosceles triangle.

6. Omitted.

7. In the sides **AB, BC, CA** of an equilateral triangle **ABC**, points **P, Q, R** are taken, so that **AP**=**BQ**=**CR**. Prove that **PQR** is an equilateral triangle.

8. In a quad. **ABCD, AB**=**AD** and **BC**=**CD**; prove that one of the diagonals bisects the other at right angles.

9. **ABC, DBC** are two isosceles triangles on the same base **BC**; prove that **AD**, or **AD** produced, bisects **BC** perpendicularly, and also bisects the $\angle$s **BAC, BDC**.

10. If the angles **B** and **C** at the base of an isosceles triangle **ABC** be bisected by **BD, CD**, shew that **DBC** is an isosceles triangle and that **AD** bisects the angle **A**.

11. If the exterior angles formed by producing the equal sides **AB, AC** of an isosceles triangle **ABC** be bisected by **BD, CD**, shew that **DBC** is an isosceles triangle and that **AD** bisects **BC** at right angles.

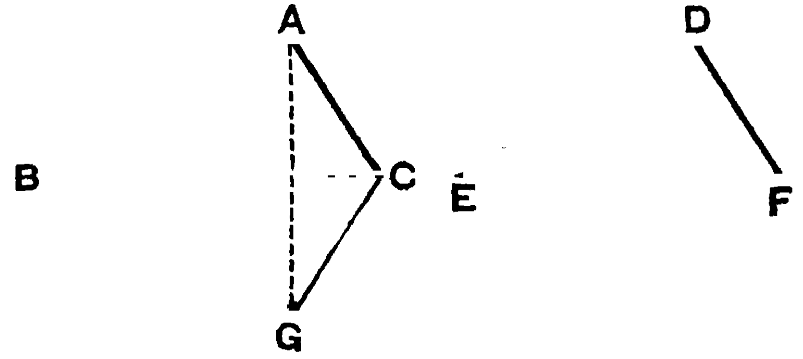
12. **ABCD** is a quadrilateral in which the $\angle$**ABC**=the $\angle$**BAD** and the $\angle$**ADC**=the $\angle$**BCD**. Prove that **AD**=**BC**.

13. and 14. Omitted.

15. **If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.**

THEOREM 7. [Euc. I. 8.]

If the three sides of one triangle are respectively equal to the three sides of another, the two triangles are congruent.



Let ABC, DEF be two triangles, in which

(i) AB = DE,
(ii) BC = EF,
(iii) AC = DF.

To prove that *the* △s ABC, DEF *are congruent.*

Let BC be the *greatest* side of the △ ABC.

Proof. Apply the △ DEF to the △ ABC, so that E falls on B and EF coincides with the equal side BC, and D falls at some point G on the side of BC opposite to A.

Then GBC is the new position of the △ DEF;
so that GB = DE, GC = DF, and the ∠ BGC = the ∠ EDF.

Join AG.

Because BA = BG, [*Hyp.*
∴ the ∠ BAG = the ∠ BGA;
and because CA = CG, [*Hyp.*
∴ the ∠ CAG = the ∠ CGA;
∴ the whole ∠ BAC = the whole ∠ BGC,
that is, the ∠ BAC = the ∠ EDF.

Then in the $\triangle^s$ **ABC, DEF**,

(i) $AB = DE$,

(ii) $AC = DF$,

and (iii) the included $\angle BAC$ = the included $\angle EDF$;

$\therefore$ the $\triangle^s$ **ABC, DEF** are congruent. [I. 4.

Q. E. D.

Obs. The triangles are congruent; hence the $\angle A$ = the $\angle D$, the $\angle B$ = the $\angle E$, the $\angle C$ = the $\angle F$.

[That is, *the angles opposite to the equal sides are equal.*]

Also the $\triangle ABC$ = the $\triangle DEF$ in area.

Note. We have chosen for superposition the *greatest side* of the $\triangle$**ABC**, in order to *avoid two other cases.* For (i) if the angle at **C** were a right angle, **AG** would coincide with **AC**, **GC** as in Fig. 1, and (ii) if the angle were an obtuse angle, then **AG** would fall outside the angles **BAC**, **BGC** as in Fig. 2.

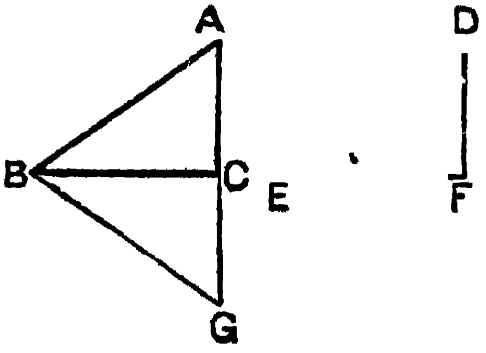


Fig. 1.

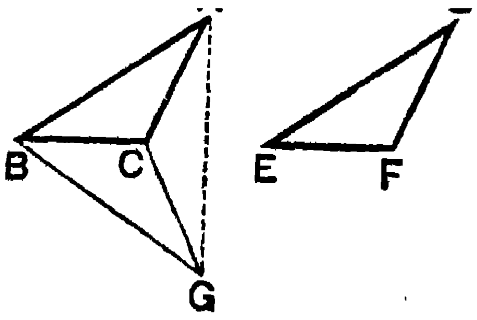


Fig. 2.

EXERCISES VII.

1. **If the opposite sides of a quadrilateral are equal, its opposite angles are equal.**

2. **The straight line joining the vertex of an isosceles triangle to the middle point of the base, bisects the vertical angle and is perpendicular to the base.**

3. Two circles whose centres are **A** and **B** intersect at **P** and **Q**; shew that the angles **APB, AQB** are equal.

4. In Exercise 3, shew that **AB** bisects **PQ** at right angles.

5. **If the perpendiculars at the middle points of any two sides of a triangle meet at a point, then the perpendicular at the middle of the third side also meets at that point.**

6. **The diagonals of a rhombus bisect its angles, and also bisect each other at right angles.**

7. **ABCD** is a quadrilateral in which **AB**=**AD**, and the ∠**ABC** =the ∠**ADC**; prove that **AC** bisects **BD** at right angles.

8. Omitted.

9. If a quadrilateral has a pair of opposite sides equal and the diagonals equal, it has two pairs of equal angles.

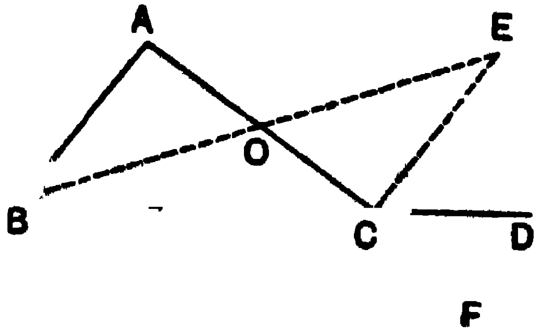
10. In a quadrilateral **ABCD, AB**=**CD** and the ∠**ABC**=the ∠**BCD**; prove that the ∠**CDA**=the ∠**DAB**.

11. **ABC, ABD** are two triangles on the same base **AB** and on the same side of it; **AC**=**BD, AD**=**BC** and **AD, BC** meet in **O**. Prove that the triangles **AOC, BOD** are equal in all respects.

12. **ABCD** is a quadrilateral in which **AB**=**BC, AD**=**DC**; **O** is a point within **ABCD** such that **AO**=**OC** shew that **BO, OD** are in the same straight line.

THEOREM 8. [Euc. I. 16.]

If one side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.



Let ABC be a triangle and let one side BC be produced to D.

To prove that *the exterior angle* ACD *is greater than either of the interior opposite angles* BAC, ABC.

Construction. Let AC be bisected at O.

Join BO, and produce it to E, making OE equal to OB.

Join EC.

Proof. Then in the $\triangle^s$ AOB, COE,

(i) $OA = OC$, [*Cons.*

(ii) $OB = OE$,

and (iii) the included $\angle AOB =$ the included $\angle COE$,

for they are vertically opp. angles ;

$\therefore$ the triangles are congruent ; [I. 4.

so that the $\angle OAB =$ the $\angle OCE$.

But the $\angle OCD >$ the $\angle OCE$;

$\therefore$ the $\angle OCD >$ the $\angle OAB$;

that is, the $\angle ACD >$ the $\angle BAC$.

Similarly if A is joined to the middle pt. of BC, and AC is produced to F, it can be proved that

the ∠ BCF > the ∠ ABC.

But the ∠ BCF = the vertically opp. ∠ ACD ; [I. 3.

∴ the ∠ ACD > the ∠ ABC.

Hence the ∠ ACD is greater than the ∠ BAC or the ∠ ABC.

Q.E.D.

Cor. *Any two angles of a triangle are together less than two right angles.*

For the ∠ABC < the ∠ACD ; ∴ the ∠ABC + the ∠ACB < the ∠ACD + the ∠ACB. But the ∠ACD + the ∠ACB = 2 right angles ;

∴ the ∠ABC + the ∠ACB < 2 right angles.

EXERCISES VIII.

1. Prove the Corollary by taking any point in one of the sides and joining it to the opposite vertex.

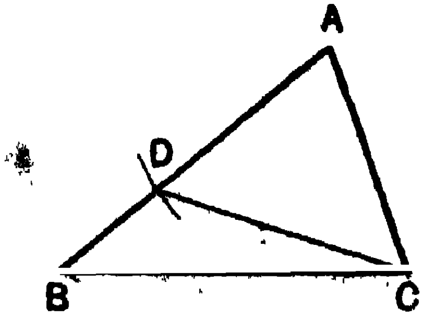
2. **In a triangle, the angle subtended by a side at any point within the triangle is greater than the angle opposite to that side.**

3. ABC is a triangle, and the angle A is bisected by AD which meets BC in D ; prove that the ∠ADB > the ∠DAB.

4. The right angle is the greatest angle of a right-angled triangle.

THEOREM 9. [Euc. I. 18.]

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.



Let ABC be a triangle, in which AB is greater than AC.

To prove that *the* ∠ACB *is greater than the* ∠ABC.

Construction. From AB, cut off AD equal to AC.
Join CD.

Proof. In the △ADC, because AC = AD ; [*Const.*
∴ the ∠ACD = the ∠ADC. [I. 5.

Now the ext. ∠ADC is greater than the interior opposite angle DBC or ABC ; [I. 8.

∴ the ∠ACD is greater than the ∠ABC.

But the ∠ACB is greater than the ∠ACD ;

∴ the ∠ACB is still greater than the ∠ABC.

Q.E.D.

EXERCISES IX.

1. **ABC** is an equilateral triangle; any point **D** is taken in **BC** and **AD** is joined. Prove that the angle **ADB** is greater than the angle **BAD** and the angle **ADC** greater than the angle **DAC**.

2. **ABC** is an isosceles triangle having **AB = AC**; **AC** is produced to **D**; prove that the angle **ADB** is less than the angle **ABD**.

3. In any triangle, the greatest side makes acute angles with each of the other two sides.

4. An angle of a triangle is acute, right, or obtuse, according as the median that bisects the opposite side is greater than, equal to, or less than, half that side. [See (XIV), 3].

5. **ABCD** is a quadrilateral of which **AD** is the longest side and **BC** the shortest; shew that the angle **ABC** is greater than the angle **ADC**, and the angle **BCD** greater than the angle **BAD**.

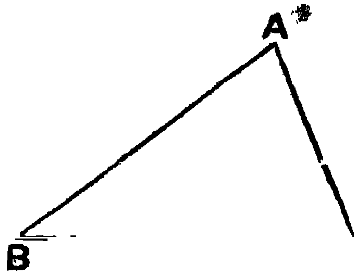
6. **ABCD** is a quadrilateral in which the angles **ABC**, **BCD** are equal; prove that the angle **ADC** is greater than, equal to, or less than the angle **BAD**, according as **AB** is greater than, equal to, or less than **CD**.

7. Prove the Theorem from the following construction:—Bisect the angle **A** by **AD** meeting **BC** in **D**; from **AB**, cut off **AE** equal to **AC**; join **DE**.

THEOREM 10. [Euc. I. 19.]

[Converse of Theorem 9.]

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.



Let ABC be a triangle, in which the ∠ACB is greater than the ∠ABC.

To prove that AB *is greater than* AC.

Proof. If AB is not greater than AC,

then either (i) AB = AC,

or (ii) AB is less than AC.

(i) If AB were equal to AC,

then the ∠ABC would be equal to the ∠ACB ; [I. 5.

but it is not, by hypothesis.

(ii) If AB were less than AC,

then the ∠ACB would be less than the ∠ABC ; [I. 9.

but it is not, by hypothesis.

Thus AB is neither equal to, nor less than, AC ;

∴ AB is greater than AC.

Q.E.D.

EXERCISES X.

1. The hypotenuse is the greatest side of a right-angled triangle.

2. The diagonal of a square is greater than its side.

3. **ABC** is a triangle whose vertical angle **A** is bisected by a straight line meeting **BC** in **D**; prove that **AB** is greater than **BD**, and **AC** greater than **CD**.

4. **ABC** is a triangle in which **AB** is greater than **AC**; **BD**, **CD** bisect the angles **ABC**, **ACB**, meeting at **D**; prove that **BD** is greater than **CD**.

5. If any point be taken in the base of a triangle, the straight line joining it to the vertex is less than the greater of the two sides.

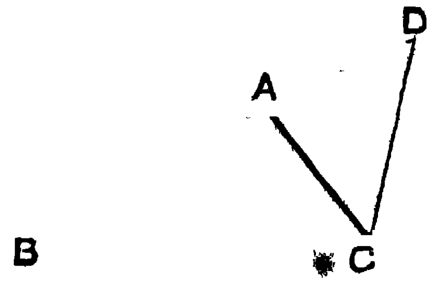
6. **A straight line drawn from the vertex of an isosceles triangle to any point in the base is less or greater than either of the equal sides, according as the point is in the base or the base produced.**

7. From **A**, one of the angular points of a square **ABCD**, a straight line is drawn to cut **BC** and meet **DC** produced in **E**; prove that **AE** is greater than a diagonal of the square.

8. **ABC** is an isosceles triangle and **D** any point in the base **BC**; if **E** be the middle point of **AD**, prove that **AE** is less than **EB** or **EC**.

THEOREM 11. [Euc. I. 20.]

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

To prove that *any two of its sides* AB *and* AC *are together greater than the third side* BC.

Construction. Produce BA to D, making AD equal to AC. Join DC.

Proof. Because AC = AD ;

∴ the ∠ACD = the ∠ADC. [I. 5.

But the ∠BCD is greater than the ∠ACD ;

∴ the ∠BCD is greater than the ∠ADC *i.e.*, the ∠BDC.

Hence in the △DBC,

the ∠BCD is greater than the ∠BDC ;

∴ BD is greater than BC. [I. 10.

But BD = AB + AD, and AD = AC ; [*Const.*

∴ BD = AB and AC together.

Hence AB and AC are together greater than BC.

Q.E.D.

Note. This is really a self-evident Theorem ; for to go to **C** from **B**, **BC** is clearly a shorter route than to go to **A** and thence to **C**. Hence *the shortest distance between two points is the straight line that joins them.*

EXERCISES XI.

1. **The difference of any two sides of a triangle is less than the third side.**

2. **Prove theorem 11, (i) by drawing a perpendicular from A to BC and (ii) by bisecting the ∠A by AD meeting BC in D.**

3. **If from the ends of a side of a triangle, two straight lines are drawn to a point within the triangle, these straight lines are together less than the other two sides of the triangle but contain a greater angle.**

4. Any side of a quadrilateral is less than the other three sides together.

5. The sum of the distances of any point from the vertices of a triangle is greater than the semi-perimeter of the triangle.

6. The sum of the distances of any point within a triangle from its vertices is less than the perimeter of the triangle.

7. The perimeter of a quadrilateral is greater than the sum of its diagonals.

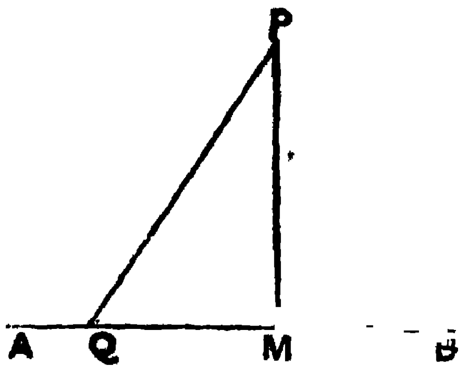
8. The sum of the diagonals of a quadrilateral is greater than the sum of either pair of opposite sides.

9. The sum of the diagonals of a quadrilateral is less than the sum of any four lines that can be drawn from any point (except the intersection of the diagonals) to the angular points.

10. **O** is any point within an equilateral triangle **ABC**. Prove that any two of the straight lines **OA, OB, OC** are together greater than the third.

THEOREM 12.

Of all straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.



Let AB be a given st. line, and P a given pt. outside it.

Let MP be the perpendicular from P to AB; and let PQ be *any* other st. line drawn from P to AB.

To prove that PM *is less than* PQ.

Proof. In the △PMQ, because PMQ is a right angle,

∴ the ∠PQM is less than a right angle; [I. 8, *Cor.*

∴ the ∠PQM is less than the ∠PMQ;

∴ PM is less than PQ. [I. 10.

Q.E.D.

Cor. *If* PM *is the shortest line from* P *to* AB, *then* PM *is perpendicular to* AB. [Converse].

For if PM is not perp. to AB, let PQ be perp. to it.

Then PQ is the shortest line and PM > PQ, which is contrary to the hypothesis.

Def. The perpendicular from a given point to a given straight line is called the **distance** of the point from the line; for it is the shortest line that can be drawn from the point to the line.

Note. The distance of an oblique from the perpendicular PM is measured either by the **angle** it makes with PM, or by the **line** joining its foot to the foot of PM. Thus the distance of PQ is either the ∠MPQ, or the line MQ. Hence PQ is nearer to PM than PR, because the ∠MPQ is less than the ∠MPR, or because MQ is less than MR.

Theorem 12 is included is the following Theorem.

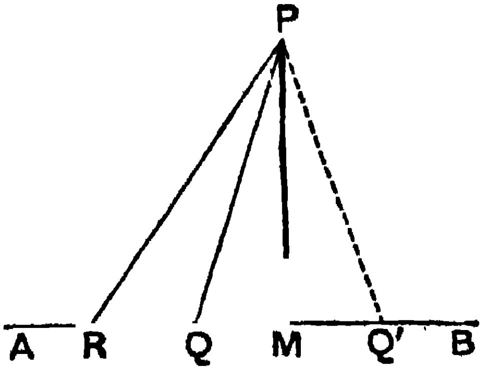
THEOREM 12A.

Of all straight lines that can be drawn to a given straight line from a given point outside it,

(i) *the perpendicular is the shortest* ;

(ii) *that which is nearer to the perpendicular is less than one more remote* ; and

(iii) *those that are at equal distances from the perpendicular are equal.*



Let AB be a given st. line, and P a given pt. outside it.

Let st. lines PM, PQ, PR,...be drawn to AB.

(i) Let PM be the perpendicular from P to AB.

To prove that PM *is the shortest of all.*

[Prove as in Theorem 12.]

(ii) Let PQ be nearer to PM than PR.

To prove that PQ *is less than* PR.

In the △PMR, since the ∠PMR is a right angle,

∴ the ∠PRM is less than a right angle. [I. 8, *Cor.*

But the ext. ∠**PQR** is greater than the ∠**PMQ** which is a right angle

∴ the ∠**PRM** is less than the ∠**PQR**,

∴ **PQ** is less than **PR**. [I 10.

(iii) Let **PQ** and **PQ′** be at equal distances from **PM** *i.e.*, let **QM** = **Q′M**.

To prove that **PQ** = **PQ′**.

In the Δˢ **PMQ**, **PMQ′**,

(1) **QM** = **Q′M**, [*Hyp.*

(2) **PM** is common ;

(3) the ∠**PMQ** = the ∠**PMQ′**, each being a rt. angle,

∴ **PQ** = **PQ′**. [I. 4.

Q.E.D.

Cor. *Two, and only two, equal straight lines can be drawn from an external point to a given straight line, one on each side of the perpendicular.*

Def. The slanting straight lines **PQ**, **PR**, **PQ′**......... are called **obliques** from the point **P** to the straight line **AB**.

EXERCISES XII.

1. **P** is a point in **AP**, the bisector of the ∠**BAC** ; if its distance from **AB** is 3·4 cm., find the distance from **AC** by measurement.

2. The distance **PQ** of a point **P** from a straight line **AB** is 1·2 cm. ; an oblique **PB** meets **AB** in **B**, such that **QB** = ·9 cm. ; find by measurement the length of **PB**.

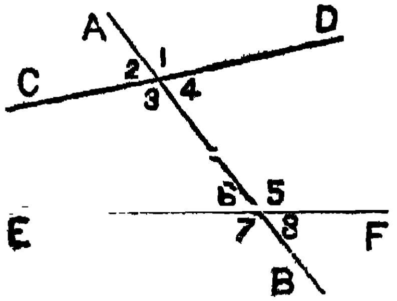
3. The point of intersection of the bisectors of any two angles of a triangle is equidistant from its sides.

4. The extremities of a straight line are equally distant from any straight line passing through its middle point.

PARALLEL STRAIGHT LINES

DEFINITIONS

1. If a straight line AB cut two other straight lines CD, EF, it makes with them eight angles, marked 1, 2, 3, 4, 5, 6, 7 and 8. Of these,



(i) 1, 2, 7 and 8, are called **exterior** angles;

(ii) 3, 4, 5 and 6, are called **interior** angles;

(iii) 3 and 5, 4 and 6, which are on *opposite* sides of AB, are called **alternate** angles.

Also 1 is the exterior angle and 5 is the **interior opposite** angle on the *same* side of AB.

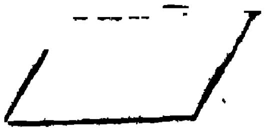
The exterior angle and the interior opposite angle on the same side of a line are called **corresponding angles.** Thus 1 and 5, 2 and 6, 3 and 7, 4 and 8, are pairs of corresponding angles.

2. A straight line drawn to cut two or more straight lines is called a **transversal** (lit. a *crossing line*).

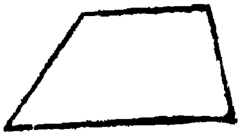
Thus in the above figure AB is a transversal.

3. **Parallel straight lines** are straight lines which being in the *same plane* do not meet however far they are produced in either direction.

4. A **parallelogram** is a four-sided figure whose opposite sides are parallel.

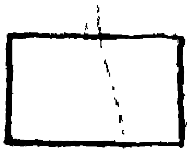


5. A **trapezium** is a four-sided figure which has only one pair of parallel sides.



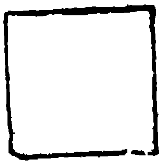
6. A **rectangle** is a parallelogram one of whose angles is a right angle.

[It will be proved later on that *all* the angles of a rectangle are right angles]

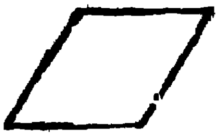


7. A **square** is a rectangle two of whose adjacent sides are equal.

[It will be shewn afterwards that *all* the sides of a square are equal and *all* its angles are right angles.]



8. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.



THEOREM 18. [Euc. I. 27 and 28].

If a straight line, cutting two other straight lines, makes—

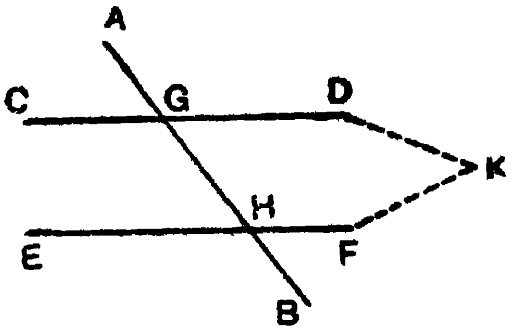
(i) the alternate angles equal ;

or (ii) two corresponding angles equal ;

or (iii) the interior angles on the same side of the line supplementary ;

then the two straight lines are parallel.

Case I



Let the st. line AB cut the st. lines CD, EF at G and H, and make the alternate ∠ˢ CGH, GHF equal.

To prove that CD and EF *are parallel.*

Proof. If CD and EF are not parallel, they will meet, if produced, either towards D and F, or towards C and E.

If possible, let CD and EF meet at K when produced towards D and F.

Then GKH is a triangle, of which one side KG is produced to C ;

∴ the ext. ∠CGH > the int. opp. ∠GHK ; [I. 8.
but this is impossible, since by hypothesis,

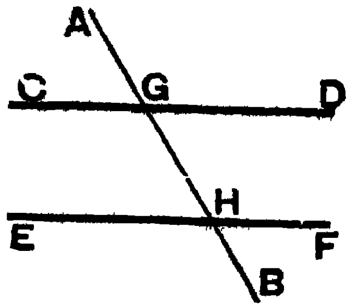
the ∠CGH = the ∠GHK.

∴ CD and EF cannot meet when produced towards D and F.

Similarly it can be proved that they cannot meet when produced towards C and E.

∴ CD and EF are parallel. Q.E.D.

Case II



Let the corresponding ∠ˢ AGD, GHF be equal.

To prove that CD *and* EF *are parallel.*

Proof. Because the ∠AGD = the vertically opp. ∠CGH ;
and the ∠AGD = the ∠GHF ; [*Hyp.*
∴ the ∠CGH = the ∠GHF ;
and these are alternate angles ;
∴ CD and EF are parallel. Q.E.D.

Case III

Let the two interior ∠ˢ DGH, GHF be supplementary.

To prove that CD *and* EF *are parallel.*

Proof. Because HG stands on the st. line CD,
∴ the ∠CGH is the supplement of the ∠DGH ; [I. 1.
and the ∠GHF is the supplement of the ∠DGH ; [*Hyp.*
∴ the ∠CGH = the ∠GHF ;
and these are alternate angles ;
∴ CD and EF are parallel. Q.E.D.

Cor. *Straight lines, which are perpendicular to the same straight line, are parallel to one another.*

Note. The student should notice that *any one* of the three conditions (i), (ii) and (iii), is sufficient to make two straight lines parallel.

Playfair's Axiom

Two intersecting straight lines cannot both be parallel to a third straight line.

Thus of the two intersecting st. lines AB and CD, if AB be parallel to EF, CD cannot be parallel to EF, or if CD be parallel to EF AB cannot be parallel to EF.

Playfair's Axiom may also be expressed thus :—

(i) *If two straight lines meet at a point and both be parallel to a third straight line, they are one and the same straight line.*

(ii) *Through a given point one, and only one, straight line can be drawn parallel to a given straight line.*

THEOREM 14. [Euc. I. 29.]

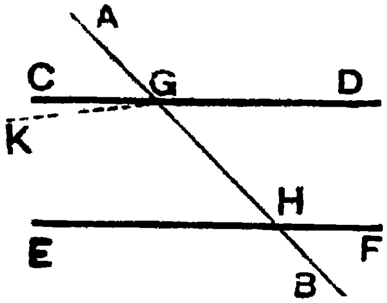
[Converse of Theorem 13.]

If a straight line cuts two parallel straight lines, it makes—

(i) the alternate angles equal;

(ii) the corresponding angles equal;

(iii) the two interior angles on the same side of the line supplementary.



Let the st. line AB cut the two parallel st. lines CD and EF at G and H.

To prove that

(i) *the* ∠CGH = *the alternate* ∠GHF;

(ii) *the* ∠AGD = *the corresponding* ∠GHF;

and (iii) *the two interior angles* DGH, GHF *are supplementary.*

(i)

Construction. If the ∠CGH be not equal to the ∠GHF, let the ∠KGH be equal to the ∠GHF, and alternate to it.

Proof. Then ∵ the ∠KGH = the alternate ∠GHF;

∴ KG is parl. to EF. [I. 13.

But CD is parl. to EF. [*Hyp.*

∴ the two interesecting st. lines CD and KG are both parl. to EF, which is impossible. [*Playfair's Axiom.*

∴ the ∠CGH is not unequal to the ∠GHF;

that is, the ∠CGH = the alternate ∠GHF.

(ii)

Again the ∠AGD = the vertically opp. ∠CGH; [I. 3.

and the ∠CGH = the alternate ∠GHF. [*Proved.*

∴ the ∠AGD = the corresponding ∠GHF.

(iii)

Lastly, because **HG** stands on **CD**.

∴ the ∠**DGH** is the supplement of the ∠**CGH** ;
and the ∠**CGH** = the alternate ∠**GHF**. [*Proved.*
∴ the ∠**DGH** is the supplement of the ∠**GHF** Q.E.D.

Note. A straight line **AB** may be drawn by moving the pencil point either from **A** to **B** or from **B** to **A**, that is, in either of two opposite directions or **senses.** Thus the line **AB** has a sense *opposite* to that of the line **BA**, the order of the letters in Geometry indicating the sense of a line.

Hence in Fig. 1, the parallels **AB**, **DE** have the *same sense* but in Fig. 2, they have *opposite senses* ; while in Fig. 3, **AC**, **DF** are drawn in the *same sense*, but **AB**, **DE** in *opposite senses.*

Cor. *The angles, whose arms are parallel, are either equal or supplementary.*

(i) The angles are **equal**, when the pairs of parallel arms are drawn *both* in the same sense or *both* in opposite senses ; as in Figs. 1 and 2.

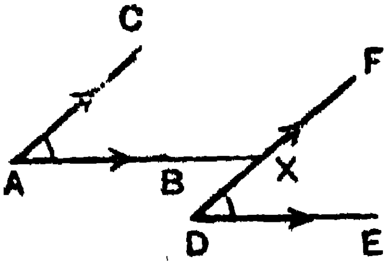


Fig. 1.

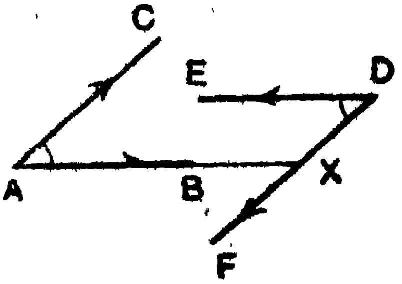


Fig. 2.

(ii) The angles are **supplementary** when *one pair* of parallel arms is drawn in the same sense and the *other pair* in opposite senses ; as in Fig. 3.

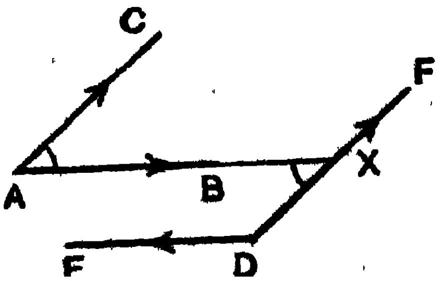
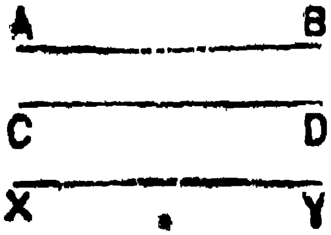


Fig. 3.

Let **A** and **D** be two ∠s whose arms are parallel.
Produce **AB** to meet **DF** at **X**.
Then in Figs. 1 and 2, the ∠**A** = the ∠**AXD** or the ∠**AXF** = the ∠**D** ; in Fig. 3, the ∠**A** = the ∠**AXD** = the supplement of the ∠**D**.

THEOREM 15. [Euc. I. 30.]

Straight lines which are parallel to the same straight line are parallel to one another.



Let the st. lines AB, CD be each parallel to the st. line XY.

To prove that AB *and* CD *are parallel.*

[First Proof]

Proof. If AB and CD are not parallel, they will intersect, if produced.

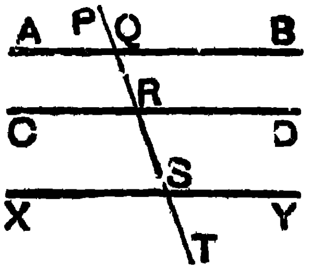
Thus two intersecting st. lines AB, CD would both be parallel to a third st. line XY, which is impossible.

[*Playfair's Axiom.*

∴ AB and CD are parallel.

Q.E.D.

[ALTERNATIVE PROOF]



Construction. Draw any st. line **PT** cutting **AB**, **CD**, **XY** at **Q**, **R** and **S**.

Proof. Since **AB** and **XY** are parallel and **PT** cuts them,

∴ the ∠AQS = the alternate ∠QSY. [I. 14.

And ∵ **CD** and **XY** are parallel and **PT** cuts them,

∴ the ∠QRD = the corresponding ∠QSY. [I. 14.

∴ the ∠AQR = the ∠QRD ;

and these are alternate angles ;

∴ **AB** and **CD** are parallel. [I. 13.

Q.E.D.

Note. This Theorem is the converse of Playfair's Axiom.

EXERCISES XIII.

1. In the diagram of Theor. 14, if the ∠**AGC** = 46°, write down the values of all the other angles made by **AB** on **CD**, **EF**.

2. In the same diagram, if the ∠**AGD** is four times the ∠**GHE**, find all the other angles.

3. **ABCD** is a parallelogram ; if the ∠**A** = 112°, find the remaining angles.

4. **AB**, **CD** are the parallel sides of a trapezium **ABCD** ; the ∠**A** is bisected by **AE**, meeting **CD** in **E** ; if the ∠**D** is $\frac{3}{5}$ of a right angle, find the ∠**AED**.

5. **ABC** is a triangle; through **A**, **PAQ** is drawn parallel to **BC**, **P** and **B** being on the same side of **AC**; if the ∠**BAP**=70° and the ∠**CAQ**=60°, find the angles of the triangle **ABC** and shew that their sum is two right angles.

6. **ABC** is a triangle of which the ∠**A** is double of the ∠**B**; through a point **P** in **BC**, **PQ** is drawn parallel to **AC** and through **A**, **AQ** is drawn parallel to **BC**, meeting **PQ** in **Q**; if the ∠**Q**=51°, find the magnitudes of the angles of **ABC**.

7. If two or more straight lines are parallel, any straight line perpendicular to one of them is also perpendicular to all the others.

8. Straight lines which are perpendicular to two intersecting straight lines cannot be parallel to one another. [See Cor. to Theorem 13.]

9. If one angle of a parallelogram is a right angle, all its angles are right angles.

10. The opposite angles of a parallelogram are equal.

11. **Any straight line parallel to the base of a triangle cuts off from it a triangle equiangular to it.**

11 (*a*). **The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.**

11 (*b*). **If the straight line bisecting an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.**

12. If the corresponding angles made by a transversal on two parallel straight lines are bisected, the bisectors are parallel.

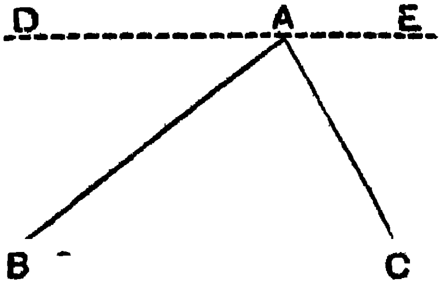
13. The bisectors of the angles made by a transversal on two parallel straight lines form a rectangle.

14. The parallels **AB**, **CD** are met by a transversal **EACF**; **AG**, **CH** bisect the angles **EAB**, **FCD** respectively; prove that the angles **EAG** and **FCH** are complementary.

15. If one of the two exterior angles which a transversal makes with two parallel straight lines on the same side of it is bisected, the perpendicular to the bisector drawn from the other exterior angle, when produced, bisects this angle.

THEOREM 16. [Euc. I. 32.]

The sum of the three angles of a triangle is equal to two right angles.



Let ABC be a triangle.

To prove that

the $\angle$ ABC + *the* $\angle$ BCA + *the* $\angle$ CAB = 2 *right angles.*

Construction. Through A, suppose DAE is drawn parallel to BC.

Proof. Because DAE is parallel to BC, and AB and AC meet them,

∴ the $\angle$ ABC = the alt. $\angle$ BAD,
and the $\angle$ BCA = the alt. $\angle$ CAE; [I. 14.

∴ the $\angle$ ABC + the $\angle$ BCA = the $\angle$ BAD + the $\angle$ CAE.

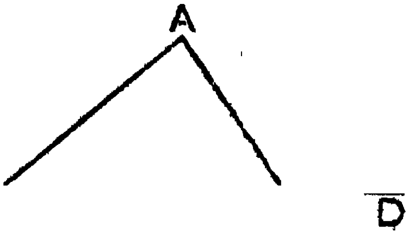
To each of these equals, add the $\angle$ CAB;

∴ the $\angle$ ABC + the $\angle$ BCA + the $\angle$ CAB
= the $\angle$ BAD + the $\angle$ CAE + the $\angle$ CAB
= 2 right angles. [I. 1.

Q.E.D.

This proof is due to the Greek philosopher PYTHAGORAS.

Cor. 1. *If a side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.*



For if BC is produced to D, then the $\angle$ ACB is the supplement of the $\angle$ ACD.

Also the $\angle$ ACB is the supplement of the $\angle$s ABC, BAC.

∴ the $\angle$ ACD = the $\angle$s ABC, BAC.

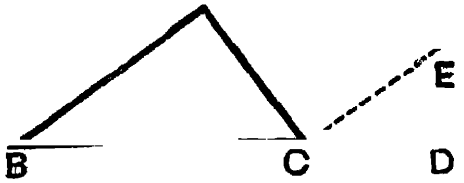
Cor. 2. *If two angles of one triangle are equal to two angles of another, their third angles are equal.*

Cor. 3. *If one angle of a triangle is equal to the sum of the other two, that angle is a right angle.*

The following is Euclid's Enunciation and Proof of this important theorem.

THEOREM 16. [ALTERNATIVE PROOF]

If a side of a triangle is produced, (i) the exterior angle is equal to the two interior opposite angles; and (ii) the three interior angles of a triangle are together equal to two right angles.



Let ABC be a triangle, and let a side BC be produced to D.

To prove that

(i) *the ext.* ∠ ACD = *the sum of the two int. opp. angles* A *and* B ;

and (ii) *the sum of the angles* A, B *and* ACB *is equal to two right angles.*

Construction. Through C, suppose CE is drawn ∥ to BA.

Proof. (i) Because BA is ∥ to CE and AC meets them;

∴ the ∠ ACE = the alternate ∠ A. [I. 14.

Also because BA is parallel to CE and BD cuts them;

∴ the ∠ DCE = the corresponding ∠ B. [I. 14.

∴ the whole ∠ ACD = the ∠ A + the ∠ B.

(ii) Again the ∠ ACD = the ∠ A + the ∠ B. [*Proved.*

To each of these equals, add the ∠ ACB;

∴ the ∠ ACD + the ∠ ACB

= the ∠ A + the ∠ B + the ∠ ACB.

But **AC** stands on the st. line **BCD** ;

∴ the ∠**ACD** + the ∠**ACB** = 2 right angles. [I. 1.

∴ the ∠**A** + the ∠**B** + the ∠**ACB** = 2 right angles.

Q.E.D.

EXERCISES XIV.

(*Numerical*)

1. In a triangle **ABC**, ∠**A** = 54° and ∠**B** = 65° ; find ∠**C**.

2. In a right-angled triangle, one of the acute angles is 35° ; find the other.

3. Find the magnitude of each of the angles of an equilateral triangle.

4. If the vertical angle of a triangle is 18° and the difference of the angles at the base 12°, find the angles at the base.

5. Each of the angles at the base of a triangle is double of the vertical angle ; express its angles in degrees.

6. The angles of a triangle are as the numbers 1, 2 and 3 ; shew that the triangle is right-angled.

7. If two angles of a triangle are equal and the third angle is less than either by 21°, find them.

8. One of the acute angles of a right-angled triangle is double of the other ; find their magnitudes.

(*Theoretical*)

1. Two at least of the angles of a triangle must be acute.

2. The two acute angles of a right-angled triangle are complementary.

3. **An angle of a triangle is obtuse, right or acute, according as it is greater than, equal to, or less than, the other two angles taken together.**

4. **One, and only one, perpendicular can be drawn to a given straight line from a given point without it.**

5. **The sum of the interior angles of any quadrilateral is equal to four right angles.**

6. The sides **AB**, **DC** of a quadrilateral **ABCD** are produced through **B** and **C** to meet. Prove that the exterior angles at **B** and **C** are together equal to the interior angles at **A** and **D**.

7. If from the extremities of the base of a triangle perpendiculars are drawn to its sides, then the vertical angle is equal to the sum of the angles made by the perpendiculars with the base.

8. **The acute angle between two perpendiculars is equal to the acute angle between the straight lines to which they are perpendicular.**

9. If the three sides of one triangle be respectively perpendiculars to those of another triangle, the triangles are equiangular.

10. **The perpendicular drawn from the right angle of a right-angled triangle to the hypotenuse, divides the triangle into two triangles which are equiangular to each other and to the whole**

11. The bisectors of any two adjacent angles of a parallelogram are at right angles.

12. **AP, BQ, CR** are drawn within the triangle **ABC**, making the angles **BAP, CBQ, ACR**, all equal to one another. If these lines do not meet in a point, prove that the triangle formed by them is equiangular to the triangle **ABC**.

13. **ABC** is a triangle, the angles at whose base **BC** are equal. If these angles are bisected by **BO, CO**, prove that the angle **BOC** is equal to an exterior angle made by producing **BC**.

14. If the base of any triangle be produced both ways, the sum of the two exterior angles diminished by the vertical angle is two right angles.

15. The bisectors of any two adjacent angles of a quadrilateral contain an angle equal to half the sum of the remaining angles.

16. The angle between the internal bisector of one of the base angles of a triangle and the external bisector of the other base angle

= *half the vertical angle.*

17. The angle between the *internal* bisectors of the base angles of a triangle

= *one right angle + half the vertical angle.*

18. The angle between the *external* bisectors of the base angles of a triangle

= *one right angle − half the vertical angle.*

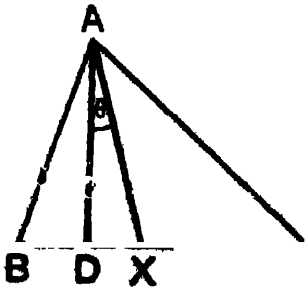
19. **The angle formed by two straight lines joining any point on a semicircle to the extremities of the diameter is a right angle.**

20. **The angle between the bisector of the vertical angle of a triangle and the perpendicular from that angle to the base is equal to half the difference of the angles at the base.**

Let **ABC** be a $\triangle$ in which the $\angle$**B** > the $\angle$**C**.

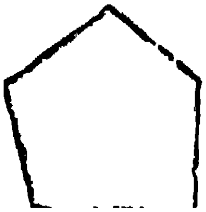
Let **AX** be the bisector of the $\angle$**A** and **AD** perp. to **BC**; also let θ (theta) = the $\angle$**DAX**.

To prove that $\theta = \frac{1}{2}(\angle B - \angle C)$.



$$\text{The } \angle \text{DAC} + \text{the } \angle \text{C} = 1 \text{ rt. } \angle$$
$$= \text{the } \angle \text{DAB} + \text{the } \angle \text{B}$$
$$\text{that is, } \tfrac{1}{2}\angle \text{A} + \theta + \text{the } \angle \text{C}$$
$$= \tfrac{1}{2}\angle \text{A} - \theta + \text{the } \angle \text{B};$$
$$\therefore 2\theta + \angle \text{C} = \angle \text{B}$$
$$\therefore 2\theta = \angle \text{B} - \angle \text{C},$$
$$\textit{i.e.}, \ \theta = \tfrac{1}{2}(\angle \text{B} - \angle \text{C}).$$

DEFINITIONS

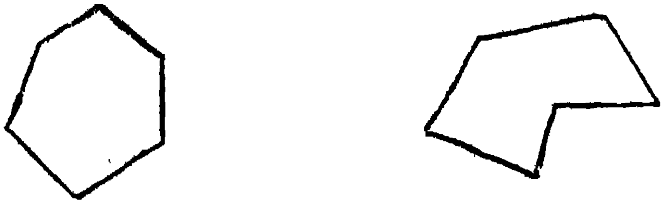


1. A **polygon** is a plane figure bounded by *more than four* straight lines.

2. A polygon is called a *pentagon*, a *hexagon*, a *heptagon*, an *octagon*, a *decagon*, a *dodecagon*, or *aquindecagon* according as it has *five*, *six*, *seven*, *eight*, *ten*, *twelve*, or *fifteen sides* respectively.

3. A **regular polygon** is a plane figure which has all its sides equal and all its angles equal.

4. A **convex polygon** is a plane figure each of whose interior angles is less than two right angles, *i.e.*, which has no re-entrant angles.



The first is a convex polygon but not the second.

THEOREM 17.

If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is equal to four right angles.

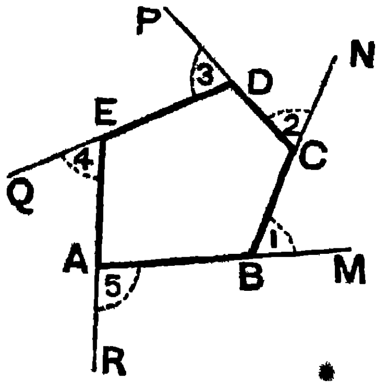


Fig. 1.

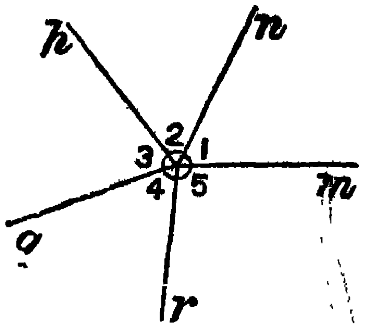


Fig. 2.

Let ABCDE be a convex polygon ; and let its sides AB, BC, CD, DE and EA be produced *in order* to M, N, P, Q and R, thus forming the ext. $\angle^s$ MBN, NCP, PDQ, QER, and RAM.

To prove that *the sum of these exterior angles* $=4$ *right angles.*

Construction. Through any pt. O, let Om, On, Op, Oq, and Or, be drawn parallel to, and in the same sense as, AB, BC, CD, DE and EA respectively.

Proof. Then because Om and On are respectively parallel to and in the same sense as, BM and BN,

∴ the $\angle$MBN = the $\angle m$On ; [I. 14. *Cor.*

i.e, the exterior angle marked 1 in fig. 1

= the angle marked 1 in fig. 2.

Similarly the angles marked with the *same numbers* in both figures are equal.

∴ the sum of all the ext. angles of the polygon

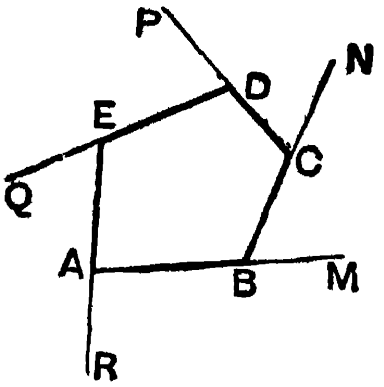
$= \angle m\text{O}n + \angle n\text{O}p + \angle p\text{O}q + \angle q\text{O}r + \angle r\text{O}m$

$=$ the sum of the angles at O $=4$ right angles.

Q.E.D.

THEOREM 18.

The sum of the interior angles of any convex polygon together with four right angles is equal to twice as many right angles as the figure has sides.



Let ABCDE be a convex polygon of n sides.

To prove that *the sum of the interior angles of the polygon* $+ 4$ *rt. angles* $= 2n$ *rt. angles.*

Construction. Produce the sides of the polygon *in order.*

Proof. Since the polygon is of n sides, therefore it has n vertices.

Now the sum of the int. and ext. angles at each vertex is 2 rt. angles.

$\therefore$ all the int. angles + all the ext. angles $= 2n$ rt. angles.

But all the ext. angles of the polygon $= 4$ rt. $\angle$s, [I. 17.

$\therefore$ all the int. angles $+ 4$ rt. $\angle$s $= 2n$ rt. angles. Q.E D.

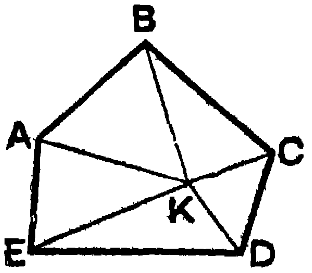
Obs. **Hence the sum of the interior angles of a polygon of n sides $= (2n-4)$ right angles.**

Again, since all the interior angles of a regular polygon are equal,

an interior angle of a regular polygon of n sides

$=$ $(2n-4)$ rt. angles

[ALTERNATIVE PROOF]



Let ABCDE be a convex polygon of n sides.

To prove that *the sum of the interior angles of the polygon + 4 rt. angles = 2n rt. angles.*

Construction. Take any pt. K within the figure, and join K to each vertex.

Proof. Then the polygon is divided into n triangles, and the sum of the angles of each triangle = 2 rt. angles.

∴ the sum of all the angles of all the $\Delta^s = 2n$ right angles.

But all the angles of all the Δ^s make up all the interior angles of the polygon, together with all the angles at K, which are 4 rt. angles. [I. 3, *Cor.* 2.

∴ the sum of the int angles of the polygon + 4 rt. angles $= 2n$ right angles. Q.E.D.

Note. The student should notice that *whatever be the number of the sides of a polygon, the sum of the exterior angles is* **constant** (being always equal to *four* right angles).

A regular polygon has all its interior angles equal; and therefore all its exterior angles are also equal. Hence an exterior angle of a regular polygon of n sides = 4 right angles ÷ n; and therefore as the number of its sides increases, an exterior angle diminishes in magnitude. Thus of two regular polygons, the one with a *greater* number of sides, has a *smaller* exterior angle and consequently a *greater* interior angle.

(i) To find an *interior* angle of a regular polygon, given the number of its sides.

We shall illustrate the method by taking a *regular hexagon.*

First Method. The figure has 6 *equal* exterior angles ;

$$\therefore \text{ an exterior angle} = \frac{360^\circ}{6} = 60^\circ ;$$

$$\text{hence an interior angle} = 180^\circ - 60^\circ = 120^\circ.$$

Second Method. All the interior angles of the figure $= 2 \times 6 - 4$ or 8 rt. angles ; $\therefore$ an interior angle $= 8 \times 90^\circ \div 6 = 120^\circ$.

(ii) To find the number of *sides* of a regular polygon, given an exterior or interior angle.

(1) Let an *exterior* angle of a regular polygon be 24°.

$$\therefore \text{ the no. of sides} = \frac{360^\circ}{24^\circ} = 15.$$

(2) Let an *interior* angle of a polygon be 150°.

Then an exterior angle $= 30^\circ$; $\therefore$ the no. of sides $= \dfrac{360^\circ}{30^\circ} = 12$.

Otherwise : Let $x =$ the no. of sides ; then $\dfrac{(2x-4)\,90^\circ}{x} = 150^\circ$; whence $x = 12$.

EXERCISES XV.

1. Three of the angles of a quadrilateral are 25°, 108° and 87° ; find the fourth angle.

2. The exterior angles formed by producing a side of a triangle both ways are 135° and 100° ; find the angles of the triangle.

3. The opposite angles of a quadrilateral are equal ; if one of them is 48°, find the others.

4. If a side of a triangle is produced and the exterior angle thus formed measures 158° and one of the interior opposite angles 37°, find each of the other angles.

5. Find the sum of the interior angles of (i) an octagon, (ii) a decagon, and (iii) a quindecagon.

6. Find the magnitude of an internal angle and of an external angle of (i) a regular pentagon, and (ii) a regular hexagon.

7. The interior angles of a rectilineal figure are together equal to 6 right angles ; how many sides has it ?

8. Can any regular polygon have its angles equal to 145° or 155° ?

9. An interior angle of a regular polygon is 120° ; how many sides has it ?

10. An exterior angle of a regular polygon is 30° ; find the number of its sides.

11. An internal angle of a regular polygon is 3 times an external angle ; how many sides has it ?

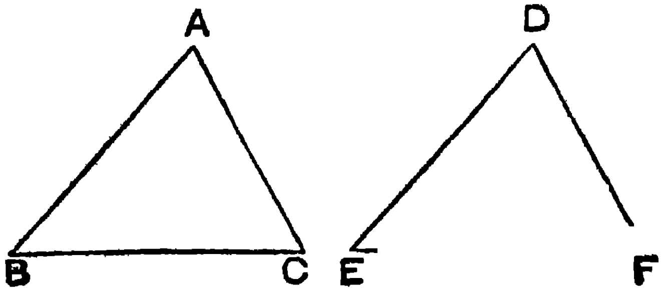
12. The interior angles of a polygon are together double of the exterior angles taken together ; what is the number of its sides ?

13. The sum of the external angles of a polygon is two-fifths of that of the internal angles ; how many sides has it ?

14. An interior angle of a regular polygon is 156° ; find the magnitude of an exterior angle of another regular polygon the number of whose sides is ⅘ of that of the first polygon.

THEOREM 19. [Euc. I. 26.]

If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another, the two triangles are congruent.



Let ABC, DEF be two triangles, in which

(i) the ∠A = the ∠D,

(ii) the ∠B = the ∠E,

and (iii) the side BC = the corresponding side EF.

To prove that *the* △^{s} ABC, DEF *are congruent.*

Proof. Since the sum of the three angles of a triangle = 2 rt. ∠^{s}, and the ∠^{s} A and B are equal to the ∠^{s} D and E;

∴ the ∠C = the ∠F.

Apply the △ABC to the △DEF, so that B falls on E and BC along EF.

Then because BC = EF;

∴ C coincides with F.

Now because the ∠B = the ∠E,

∴ BA falls along ED, and A lies in ED.

Also because the ∠C = the ∠F;

∴ CA falls along FD, and A lies in FD.

∴ the point A, which lies both in ED and FD, must coincide with D, the *only point common* to ED and FD, being the pt. of intersection.

Hence the △ ABC coincides with the △ DEF ;
that is, the △ˢ ABC, DEF are congruent. Q.E.D.

Obs. The triangles are congruent, so that

$$AB = DE \text{ and } AC = DF.$$

Also the △ ABC = the △ DEF in area.

Def. A **median** of a triangle is the straight line that joins an angular point to the middle point of the opposite side.

A triangle has thus *three* medians.

EXERCISES XVI.

* 1. **If the bisector of the vertical angle of a triangle is perpendicular to the base, the triangle is isosceles.**

2. The perpendicular to the bisector of an angle at any point cuts the arms at points which are equidistant from the vertex.

3. **The perpendiculars drawn to the arms of an angle from any point in its bisector are equal.**

4. **In a triangle, the perpendiculars drawn from the extremities of a side to its median are equal.**

5. **ACB**, **ADB** are two triangles on the same base **AB** ; if **CD** (produced, if necessary) bisects the angles **ACB**, **ADB**, prove that the triangles **ACD**, **BCD** are equal in all respects.

6. **A**, **B** are points on the opposite sides of **CD** and are such that the perpendiculars from them to **CD** are equal ; prove that **AB** is bisected by **CD**.

7. **ABCD** is a quadrilateral, and the diagonal **AC** bisects the angles at **A** and **C** ; shew that **AC** is at right angles to **BD**, the other diagonal.

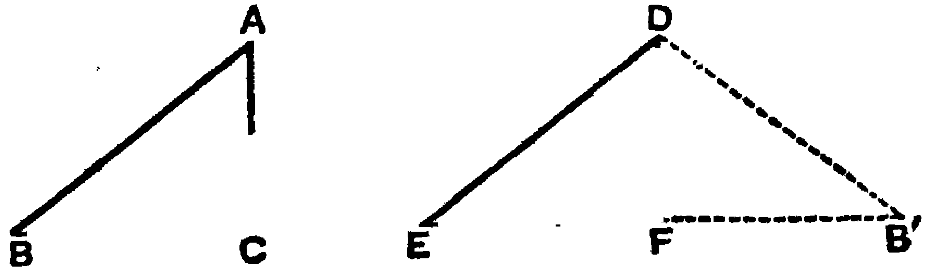
8. Through a given point, draw a straight line which shall cut off equal parts **AD**, **AE** from two intersecting straight line **AB**, **AC**.

9. **A straight line cuts two parallel straight lines at A and B. If O be the middle point of AB, prove (i) that O is equidistant from the parallels, and (ii) that any other straight line passing through O and terminated by the parallels is bisected at O.**

10. In Exercise 9, prove that any two straight lines passing through **O** cut off equal portions of the parallels.

THEOREM 20.

If two right-angled triangles have their hypotenuses equal and one side of the one equal to one side of the other, the triangles are congruent.



Let ABC, DEF be two right-angled triangles, having right angles at C and F; in which

(i) the hypotenuse AB = the hypotenuse DE,

and (ii) the side AC = the side DF.

To prove that *the* $\triangle^s$ ABC, DEF *are congruent.*

Proof. Apply the $\triangle$ABC to the $\triangle$DEF, so that A falls on D, and AC coincides with the equal side DF, and B falls at B′ on the side of DF opposite to E.

Then the $\triangle$DB′F is the $\triangle$ABC in its new position; so that the $\angle$DFB′ = the $\angle$ACB, the $\angle$B′ = the $\angle$B, and DB′ = AB.

Since each of the $\angle^s$ DFE, DFB′ is a right angle,

∴ FE and FB′ are in the same straight line. [I. 2.

Hence DEB′ is a triangle, in which DE = DB′;

∴ the $\angle$DEB′ = the $\angle$DB′E; [I. 5.

that is, the $\angle$DEF = the $\angle$ABC.

Now in the $\triangle^s$ ABC, DEF,

(i) the $\angle$ACB = the $\angle$DFE, being right angles;

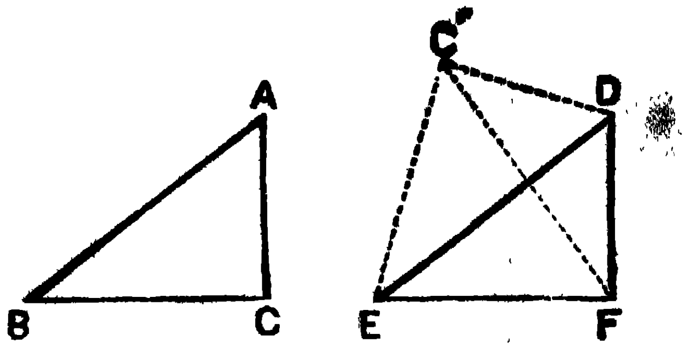
(ii) the $\angle$ABC = the $\angle$DEF, [*Proved.*

and (iii) AC = DF; [*Hyp.*

∴ the $\triangle^s$ ABC, DEF are congruent. [I. 19.

Q.E.D.

THEOREM 20. [ALTERNATIVE PROOF]



Let ABC, DEF be two right-angled triangles having right angles at C and F; in which

(i) the hypotenuse AB = the hypotenuse DE,

and (ii) the side AC = the side DF.

To prove that *the* Δ^s ABC, DEF *are congruent.*

Proof. Apply the Δ ABC to the Δ DEF,

so that A falls on D and the hypot. AB coincides with the hypot. DE (they being equal); and C falls at some pt. C′ on the side of DE opposite to F.

Then the Δ DEC′ is the Δ ABC in its new position; so that DC′ = AC, and EC′ = BC.

Join FC′.

Because DF = DC′,

∴ the ∠ DFC′ = the ∠ DC′F. [I. 5.

But the ∠ DFE = the ∠ DC′E, each being a right angle;

∴ the ∠ EFC′ = the ∠ EC′F;

∴ EF = EC′ = BC.

Then in the Δ^s ABC, DEF,

(i) AB = DE, (ii) AC = DF, } [*Hyp.*

(iii) BC = EF; [*Proved.*

∴ the Δ^s ABC, DEF are congruent. [I. 7.

Q.E.D.

EXERCISES XVII.

1. **The perpendicular from the vertex of an isosceles triangle bisects the base as well as the vertical angle.**

2. The perpendiculars drawn from the vertices of an equilateral triangle to the opposite sides are equal to one another.

3. If the perpendiculars from two of the angular points of a triangle to the opposite sides are equal, the triangle is isosceles.

4. **ABC** is an angle whose arms **AB**, **BC** are equal; **DO**, **EO** are perpendiculars to **AB**, **BC** at their middle points, meeting at **O**; prove that **OD**=**OE**.

5. **P** is a point within the angle **AOB**, such that the perpendiculars from **P** to **OA**, **OB** are equal; prove that **OP** bisects the angle **AOB**.

6. If **O** be a point within the triangle **ABC** such that the perpendiculars from it to the sides are equal, prove that **OA**, **OB**, **OC** bisect the angles **A**, **B**, **C**.

7. The bisectors of the angles **ABC** and **ACB** of a triangle **ABC** meet at **I**; shew that **AI** bisects the angle **BAC**.

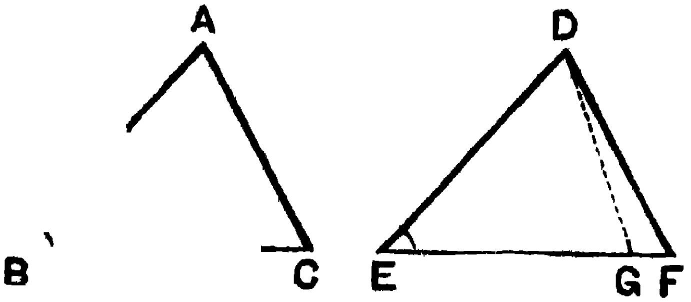
8. The bisectors of the exterior angles formed by producing the sides **AB**, **AC** of a triangle **ABC** meet at **I**; shew that **AI** bisects the angle **BAC**.

9. **If two triangles have two sides of the one equal to two sides of the other and the perpendicular to one of these sides from the opposite angle equal to the perpendicular to the corresponding side of the other, the two triangles are equal in all respects.**

10. **Q** is a point on the perpendicular drawn from **A** to the opposite side **BC** of a triangle; if **QB**=**QC**, prove that the triangle is isosceles.

* THEOREM 21.

If two triangles have two sides of the one respectively equal to two sides of the other, and the angles opposite to a pair of equal sides equal; then if the angles opposite to the other pair of equal sides are of the same species (*i.e.*, **both acute, or both obtuse), the two triangles are congruent.**



Let ABC, DEF be two triangles, in which

(i) AB = DE, (ii) AC = DF, (iii) the ∠ B = the ∠ E, and (iv) the ∠ˢ C and F are of the same species.

To prove that *the* Δˢ ABC, DEF *are congruent.*

First, let the ∠ˢ C and F be both *acute.*

Proof. If the ∠ BAC = the ∠ EDF,

then the Δ ABC ≡ the Δ DEF.

But if the ∠ BAC be not equal to the ∠ EDF, make the ∠ EDG equal to the ∠ BAC.

Let DG meet EF, or EF produced, in G.

Then in the Δˢ ABC, DEG,

(i) the ∠ BAC = the ∠ EDG, [*Const.*

(ii) the ∠ B = the ∠ E, } [*Hyp.*
and (iii) AB = DE; }

∴ the Δ ABC ≡ Δ DEG; [I. 19.

so that AC = DG, and the ∠ C = the ∠ DGE.

But the ∠ C is acute (*Hyp.*); ∴ the ∠ DGE is acute.

Again AC = DF (*Hyp.*), and AC = DG (*Proved*) ;

∴ DF = DG ;

∴ the ∠F = the ∠DGF.

But the ∠F is acute (*Hyp.*) ; ∴ the ∠DGF is acute and consequently the ∠DGE is obtuse.

Hence the ∠DGE is both acute and obtuse, which is absurd,

∴ the ∠BAC = the ∠EDF ;

∴ the △ABC ≡ the △DEF. [I. 4.

The proof is exactly similar when the ∠ˢ C and F are both *obtuse.*

Q E.D.

Note. If each of the angles C and F be a right angle, then also the triangles are congruent. For it can be proved, as above, that each of the angles F and DGF is a rt. ∠, which is absurd, ∵ DFG is a triangle.

Of course when C and F are both right angles, the theorem can be proved as above, or by the help of I. 4, or I. 19, or I. 20.

Congruence of Triangles

Every triangle has six parts or *elements*, viz., *three* sides and *three* angles.

We have seen that two triangles are congruent, if the following three elements of the one are equal to the corresponding three elements of the other, *viz.*,

(i) *Two sides and the included angle* (I. 4) ;

(ii) *The three sides* (I. 7) ;

(iii) *Two angles and one side* (I. 19) ;

(iv) *Two sides and the angle opposite to a pair of equal side a right angle* (I. 20) ;

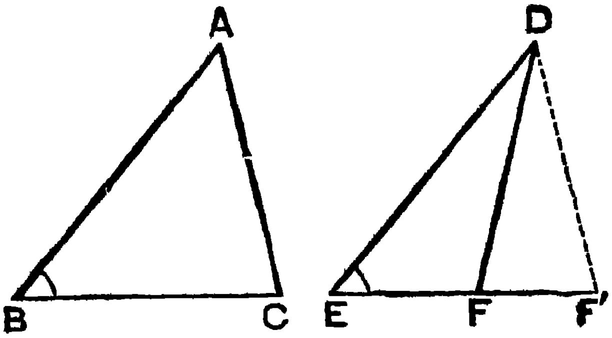
(v) *Two sides and one angle opposite to a pair of equal sides*, [if the angles opposite to the other pair are of the same species] (I. 21).

There are however two cases in which *three* elements being equal, the triangles are *not* congruent :—(i) *Three angles* and (ii) *Two sides and an angle* opposite to a pair of equal sides.

In the first case, in fact, only *two* parts are given equal and not *three* ; for if *two* angles of one triangle are equal to *two* angles of another, the *third* angles must necessarily be equal.

In the second case also the triangles are not necessarily congruent.

Let ABC, DEF be two $\triangle^s$ in which AB = DE, AC = DF, and the $\angle$ABC = the $\angle$DEF.



If the $\angle$BAC be not equal to the $\angle$EDF, then at the pt. D, on the same side of DE as F, make the $\angle$EDF′ equal to the $\angle$BAC, and let EF, produced if necessary, cut DF′ at F′.

Then in the $\triangle^s$ ABC, DEF′,

(i) the $\angle$ABC = the $\angle$DEF′,

(ii) the $\angle$BAC = the $\angle$EDF′,

(iii) AB = DE ;

∴ the $\triangle$ABC ≡ the $\triangle$DEF′, [I. 19.

Hence AC = DF′, and the $\angle$ACB = the $\angle$DF′E.

But AC = DF (*Hyp.*) ; ∴ DF = DF′.

Thus the $\triangle$ABC has three elements equal to the corresponding three elements of the *two* triangles DEF and DEF′ one of which the $\triangle$DEF′ ≡ the $\triangle$ABC, while the other, *viz.*, the $\triangle$DEF, is not.

Again ∵ AC = DF′ (*proved*) = DF ;

∴ the ∠DFF′ = the ∠DF′F = the ∠ACD (*proved*).

But the ∠DFE is the supplement of the ∠DFF′ ;

∴ the ∠DFE is the supplement of the ∠ACB.

Hence *if two triangles have two sides of the one respectively equal to two sides of the other and the angles oppsite to a pair of equal sides equal, then the angles opposite to the other pair of equal sides are either* ***equal*** (as in the Δˢ ABC, DEF′) or ***supplementary*** (as in the Δˢ ABC, DEF).

EXERCISES XVIII.

1. **The angle between the bisector of an angle and the line that divides the angle into two unequal angles, is half the difference between the unequal angles.**

2. If **ABC** is a triangle such that **D**, the middle point of **AB** being joined to **C**, **DC = DA** ; then the ∠**ACB** is a right angle.

3. **ABC** is an isosceles triangle, having **AB = AC** ; **BA** is produced to **D**, so that **AD = AB** ; if **DC** is joined, prove that the angle **BCD** is a right angle.

4. **ABC** is an isosceles triangle, whose base is **BC** ; from any point **D** in **BC**, a perpendicular is drawn to **BC**, meeting **AC** in **E** and **BA** produced in **F** ; prove that the triangle **AEF** is isosceles.

5. **BC** is the base of an isosceles triangle ; the ∠**ABC** is bisected by **BD** which meets **AC** at **D** ; and **DE** is drawn parallel to **BC** to meet **AB** at **E** ; shew that **CD, DE, EB** are all equal to one another.

6. **In a right-angled triangle, the straight line joining the middle point of the hypotenuse to the right angle is half the hypotenuse.**

7. In a right-angled triangle, if one of the acute angles is double of the other, then the side adjacent to it is half the hypotenuse.

[Observe that the angle is $\frac{2}{3}$ of a right angle, *i.e.*, 60°.]

8. A straight line **AB** is bisected at **C**, and on the same side of the line two triangles **CAD, CBE** are described, having the two sides **CD, DA** equal to the two sides **CE, EB**, each to each. Prove that the lines **AE, BD** are equal.

9. ***ABCDEF*** *is a regular* hexagon; prove that **AB** is parallel to **ED.**

10. **ABC** is a triangle; **D**, **E** are the middle points of **BC** and **AB** respectively; **AD, CE** are produced to **F** and **G**, so that **AD**=**DF** and **CE**=**EG**; prove that **FB** and **GB** are in the same straight line.

11. Through the middle point **D** of the side **AB** of a triangle **ABC**, **DE** is drawn parallel to **BC** meeting **BE**, the bisector of the angle **ABC** in **E**. Prove that **AEB** is a right angle.

12. Perpendiculars, drawn from the ends of the base of an isosceles triangle to the opposite sides, are equal and make equal angles with the base, each of these angles being half the vertical angle.

13. **ABC** is an isosceles triangle; **D** is any point in the base **BC**; **DE, DF** are drawn parallel to **AB**, **AC**, meeting the sides in **E, F**. Shew that the perimeter of the parallelogram **AEDF** is constant.

14. **ABCD** is a quadrilateral, having **AB** parallel to **CD**; the angle **A** is bisected by **AE**, meeting **CD** in **E**; shew that the triangle **ADE** is isosceles.

15. **ABC** is a triangle; **AD, AE** are drawn from **A** to **BC**, making the ∠ **BAD**=the ∠ **C** and the ∠ **CAE**=the ∠ **B**; prove that the perpendicular from **A** to **BC** bisects **DE.**

16. If one angle of a triangle is double of another, an isosceles triangle may be added to it, so that they two together will form a single isosceles triangle.

17. **AB** is a straight line; **PC** meets it in **C**; **CQ, CR** bisect the angles **ACP**, **BCP** respectively; if **QPR** be drawn parallel to **AB**, prove that it is bisected by **CP.**

18. **ABC** is an isosceles triangle, having **AB**=**AC**; **D** is any point in **AC**; **AB** is produced to **E**, so that **BE**=**CD**; prove that **DE** is bisected by **BC.**

19 **ABC** is an isosceles triangle whose base is **BC**; with centre **C**, a circle is described passing through **B** and cutting **AB, AC** in **D, E** respectively; prove that **DE** is parallel to the bisector of the angle **ABC**.

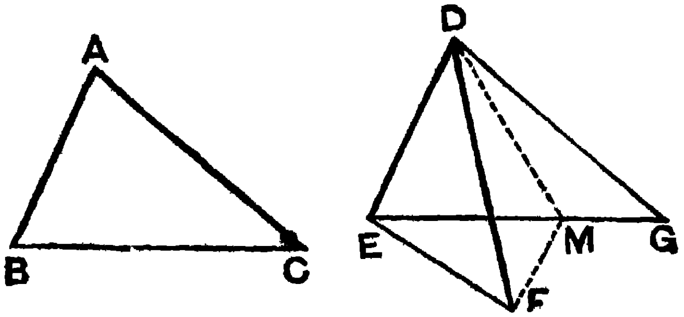
20. How many equiangular triangles can be placed so as to have one common angular point and fill up the space round it?

21. The angles of a hexagon are as the numbers 10, 15, 22, 30, 32 and 35; find the magnitude of each in degrees.

22. There are two regular polygons; the number of sides of one is double the number of sides of the other, and an angle of one polygon is to an angle of the other as 5 is to 4; find the number of sides of each.

THEOREM 22. [Euc. I. 24.]

If two triangles have two sides of the one respectively equal to two sides of the other but the included angles unequal, then the base of that which has the greater included angle is greater than the base of the other.



Let ABC, DEF be two triangles, in which

(i) AB = DE,

(ii) AC = DF,

and (iii) the ∠BAC is greater than the ∠EDF.

To prove that BC *is greater than* EF.

Proof. Apply the △ABC to the △DEF, so that A falls on D and AB coincides with the equal side DE.

Then because the ∠BAC is greater than the ∠EDF,

∴ AC falls outside the ∠EDF.

Let DG, EG represent AC, BC in their new position.

Let DM be drawn bisecting the ∠FDG to meet EG in M.

Join FM.

Then in the $\triangle^s$ FDM, GDM,

(i) DF = DG,

(ii) DM is common,

and (iii) the included ∠FDM = the included ∠GDM ;

∴ FM = GM. [I. 4.

Now the two sides EM, FM are greater than EF ; [I. 11.

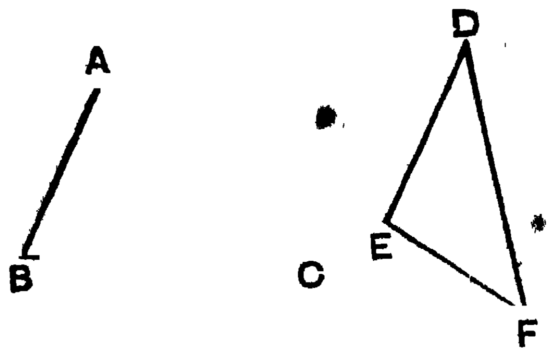
∴ EM + MG or EG is greater than EF ;

that is, BC is greater than EF. Q.E.D.

* THEOREM 22A. [Euc. I. 25.]

[Converse of Theorem 22]

If two triangles have two sides of the one respectively equal to two sides of the other but the bases unequal, then the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides of the other.



Let ABC, DEF be two triangles, in which

(i) AB = DE,

(ii) AC = DF,

and (iii) BC is greater than EF.

To prove that *the* $\angle$ BAC *is greater than the* $\angle$ EDF.

Proof. If the $\angle$ BAC is not greater than the $\angle$ EDF,

then either (i) the $\angle$ BAC = the $\angle$ EDF,

or (ii) the $\angle$ BAC is less than the $\angle$ EDF.

(i) If the $\angle$ BAC were equal to the $\angle$ EDF,

then BC would be equal to EF ; [I. 4.

which is contrary to the hypothesis.

(ii) If the $\angle$ BAC were less than the $\angle$ EDF,

then BC would be less than EF ; [I. 22.

which is contrary to the hypothesis.

Thus the $\angle$ BAC is neither equal to, nor less than the $\angle$ EDF ;

$\therefore$ the $\angle$ BAC is greater than the $\angle$ EDF.

Q.E.D.

EXERCISES XIX.

1. AB, AC, AD are three equal straight lines, of which AC is between AB and AD; prove that $BD > BC$.

2. In the side AB and the side AC produced of a triangle ABC, points D and E are taken such that $BD = CE$; prove that $BE > CD$.

3 In the sides AB, AC of a triangle ABC, in which $AB > AC$, points D, E are taken such that $BD = CE$; prove that $BE > CD$.

4. In the sides AB, AC produced of a triangle ABC, in which $AC > AB$, points D, E are taken such that $BD = CE$; prove that $BE > CD$.

5. In the sides AB, AC produced of a triangle ABC, points D, E are taken such that $BD = CE$; if $BE > CD$, prove that $AC > AB$.

6. D is the middle point of the side BC of a triangle ABC; prove that the $\angle ADB$ is obtuse, right, or acute, according as AB is greater than, equal to, or less than AC.

7. ABC is a triangle in which $AB > AC$; if P is any point in the median AD, prove that $PB > PC$.

8. Prove that the diagonals of a rhombus are unequal.

9. $ABCD$ is a quadrilateral, in which $AD = BC$ and the $\angle ADC >$ the $\angle DCB$; shew that $AC > BD$.

10. $ABCD$ is a quadrilateral, in which $AD = BC$ but $CD > AB$; shew that the $\angle CBD >$ the $\angle ADB$ and the $\angle CAD >$ the $\angle ACB$.

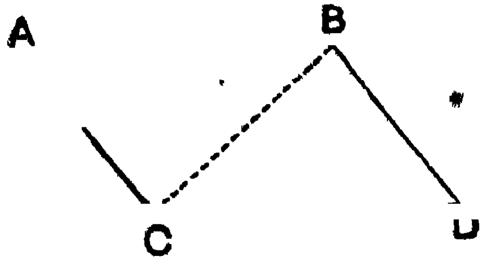
11. The angle A is the greatest angle of a triangle ABC; AO, BO are drawn making equal angles with AB; shew that the $\angle BOC >$ the $\angle AOC$.

12. AB, AC are the equal sides of an isosceles triangle ABC; P is any point within the Δ ABC such that the $\angle PBC >$ the $\angle PCB$, shew that the $\angle PAC >$ the $\angle PAB$.

PARALLELS AND PARALLELOGRAMS

* THEOREM 23. [Euc. I. 33.]

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB, CD be two equal and parallel straight lines; and let them be joined towards the same parts by AC, BD.

To prove that AC *and* BD *are equal and parallel.*

Construction. Join BC.

Proof. Because AB is parallel to CD, and BC meets them;

∴ the ∠ABC = the alternate ∠DCB. [I. 14.

Then in the Δ^s ABC, DCB,

(i) AB=CD,

(ii) BC is common to both,

and (iii) the included ∠ABC=the included ∠DCB;

∴ AC=DB,

and the ∠ACB=the ∠DBC. [I. 4.

But these are alternate angles;

∴ AC is parallel to BD. [I. 13.

Thus AC and BD are both equal and parallel. Q.E.D.

EXERCISES XX.

1. **The straight lines which join the extremities of two equal and parallel straight lines towards the opposite parts bisect each other.**

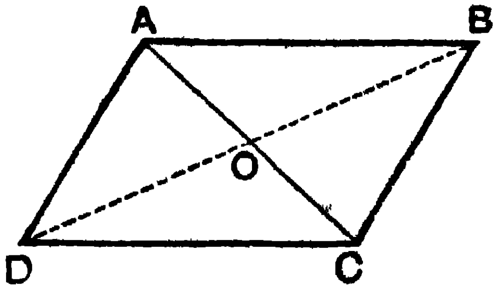
2. ABC, PQR are two triangles, such that AB, BC are respectively equal and parallel to PQ, QR; prove that AC is equal and parallel to PR.

3. A straight line which joins the extremities of two equal but not parallel straight lines and makes equal angles with them on the same side, is parallel to the straight line which joins the other extremities.

4. If a hexagon has its opposite sides equal and parallel, the three straight lines joining the opposite angles meet in a point.

THEOREM 24. [Euc. I. 34.]

The opposite sides and angles of a parallelogram are equal; each diagonal bisects the parallelogram; and the diagonals bisect one another.



Let ABCD be a parallelogram, of which AC, BD are diagonals.

To prove that

(i) AB = CD and BC = DA,
(ii) *the* ∠ABC = *the* ∠CDA,
(iii) *the* ∠BAD = *the* ∠DCB,
(iv) AC *bisects the parallelogram*,
and (v) AC and BD *bisect one another*,

Proof. Because AB and CD are parallel, and AC meets them,

∴ the ∠BAC = the alternate ∠DCA;

and because BC and DA are parallel, and AC meets them,

∴ the ∠BCA = the alternate ∠DAC. [I. 14.

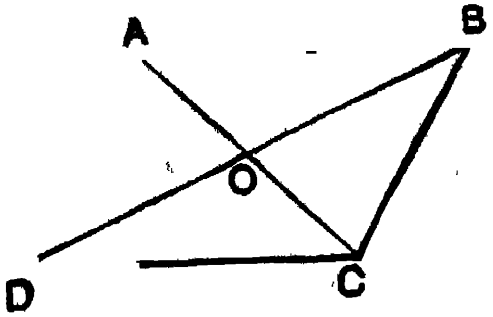
Then in the △ˢ ABC, CDA,

(1) the ∠BAC = the ∠DCA,
(2) the ∠BCA = the ∠DAC,
(3) AC is common to both;
∴ the triangles are congruent; [I. 19.
so that AB = CD and BC = DA, (i)
the ∠ABC = the ∠CDA, (ii)
and the △ˢ ABC, CDA are equal in area;

that is, AC bisects the parallelogram. (iv)

Also the ∠BAC = the ∠DCA,
and the ∠DAC = the ∠BCA, [*Proved.*
∴ the whole ∠BAD = the whole ∠DCB. (iii)

Lastly, let AC, BD intersect at O.



Then in the △s AOB, COD,
(1) the ∠OAB = the alternate ∠OCD,
(2) the ∠OBA = the alternate ∠ODC,
and (3) AB = CD ; [*Proved.*
∴ OA = OC and OB = OD ; [I. 19.
that is, AC and BD bisect one another. (v)

Q.E.D.

Cor. 1. *A rectangle has all its angles right angles.*

Let **ABCD** be a rectangle. Since a rectangle is a parm.,
∴ if the ∠**A** is a rt. angle ; the adj. ∠**B** is also a rt. angle ;
and their opposite ∠s **C** and **D** are also rt. angles.

Cor. 2. *A square has all its sides equal and all its angles right angles.*

Since a square is a rect., ∴ it has all its angles rt. angles ; and ∵ its adj. sides as well as opposite sides are equal.
∴ its four sides are equal.

Cor. 3. *Parallel straight lines are always equidistant from one another.*

EXERCISES XXI.

1. If the two adjacent sides of a parallelogram are equal, it is a rhombus.

2. The diagonals of a rectangle are equal.

3. If the diagonals of a parallelogram are equal, it is a rectangle.

4. If a diagonal of a parallelogram bisect the angles it joins, the parallelogram is a rhombus.

5. If the diagonals of a parallelogram are at right angles to each other, it is a rhombus.

6. **ABCD, ABEF** are two parallelograms having a common side **AB.** If **CE, DF** are joined, shew that **CDFE** is a parallelogram.

7. The straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

8. In a trapezium, if the oblique sides are equal, the opposite angles are supplementary; and *conversely*, if the opposite angles are supplementary, the oblique sides are equal.

9. Any straight line drawn through the point of intersection of the diagonals of a parallelogram to meet the sides is bisected at that point.

10. Every straight line drawn through the intersection of the diagonals of a parallelogram and terminated by the sides bisects the parallelogram.

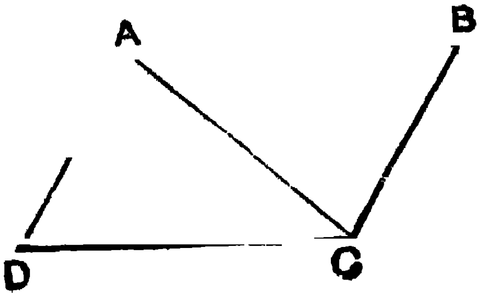
11. **P, Q** are any two points on the opposite sides of a parallelogram **ABCD**, such that **AP = CQ**. Prove that **PQ** passes through a fixed point.

12. The sum, or difference, of the perpendiculars drawn to the sides of an isosceles triangle from any point in the base, or the base produced, is equal to the perpendicular drawn from an end of the base on the opposite side, and is therefore constant.

*THEOREM 25.

[CONVERSE OF THEOREM 24.]

A quadrilateral is a parallelogram,
(i) if its opposite sides are equal,
or (ii) if its opposite angles are equal,
or (iii) if its diagonals bisect one another.



Case I

Let ABCD be a quadrilateral in which

AB = CD and BC = DA.

To prove that ABCD *is a parallelogram.*

Construction. Join AC.

Proof. In the △ˢ ABC, CDA,

(i) AB = CD,

(ii) BC = DA,

and (iii) AC is common to both ;

∴ the △ˢ are congruent ; [I. 7.

∴ the ∠BAC = the ∠DCA

and the ∠BCA = the ∠DAC ;

∴ AB is parallel to CD and BC to DA ; [I. 13.

hence ABCD is a parallelogram.

Case II

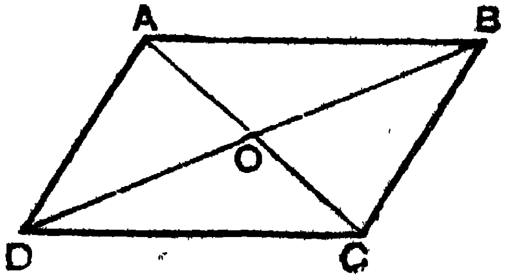
Let ABCD be a quadrilateral, in which

the ∠ABC = the ∠CDA and the ∠BAD = the ∠DCB.

To prove that ABCD *is a parallelogram.*

Proof. Since the ∠ABC = the ∠CDA
and the ∠BAD = the ∠DCB ;
∴ the ∠ABC + the ∠BAD = the ∠CDA + the ∠DCB.
But the sum of the 4 angles of a quadrilateral = 4 rt. angles ;
∴ the ∠ABC + the ∠BAD = 2 rt. angles ;
∴ AD is parallel to BC. [I. 13.

Similarly it can be shewn that AB is parallel to CD.
∴ ABCD is a parallelogram.



Case III

Let ABCD be a quadrilateral, whose diagonals AC, BD bisect one another at O.

To prove that ABCD *is a parallelogram.*

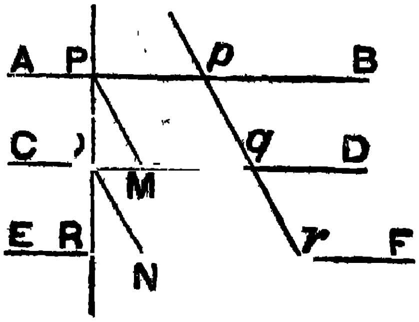
Proof. In the Δ^s AOB, COD,
(i) AO = CO,
(ii) BO = DO,
and (iii) the ∠AOB = the vertically opp. ∠COD ;
∴ the Δ^s are congruent, [I. 4.
so that the ∠OAB = the ∠OCD ;
∴ AB is parallel to CD.

Similarly it can be shewn that the Δ^s AOD, COB are congruent ; hence AD is parallel to BC.
∴ ABCD is a parallelogram,

Q.E.D.

THEOREM 26.

If there are three or more parallel straight lines, and the intercepts made by them on any transversal are equal, then the corresponding intercepts on any other transversal are also equal.



Let AB, CD, EF be parallel straight lines; and let them make equal intercepts PQ, QR on any transversal PQR; also let pq, qr be the corresponding intercepts on any other transversal pqr.

To prove that $pq = qr$.

Construction. Through P and Q, let PM, QN be drawn parallel to pqr, to meet CD and EF respectively in M and N.

Proof. Then since CD is parallel to EF, and PR meets them;

∴ the ∠PQM = the corresponding ∠QRN. [I. 14.

Also PM and QN are each parallel to pr;

∴ PM is parallel to QN; [I. 15.

∴ the ∠QPM = the corresponding ∠RQN.

Now in the Δ^s PQM, QRN,

(i) the ∠PQM = the ∠QRN. [*Proved.*

(ii) the ∠QPM = the ∠RQN, [*Proved.*

and (iii) PQ = QR; [*Hyp.*

∴ the triangles are congruent, [I. 19.

so that PM = QN.

Again because **PM**qp and **QN**rq are parallelograms ;

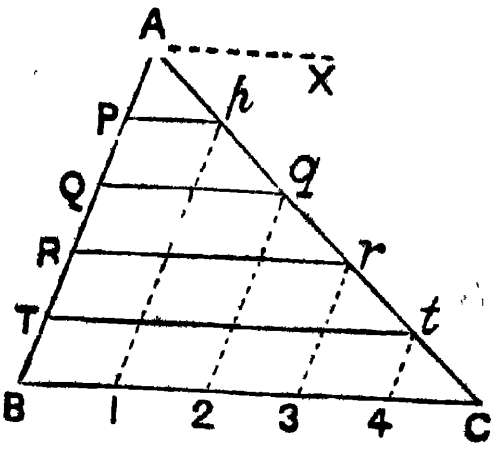
$$\left.\begin{array}{l}\therefore\ \mathrm{PM}=\text{the opp. side } pq,\\ \text{and } \mathrm{QN}=\text{the opp. side } qr\end{array}\right\}\qquad\text{[I. 24.}$$

$$\text{But } \mathrm{PM}=\mathrm{QN}\ ;\qquad [\textit{Proved.}$$

$$\therefore\ pq=qr.$$

Q.E.D.

Cor. *In a triangle, if a number of straight lines, parallel to the base, divide one side into equal parts, they also divide the other side into the same number of equal parts.*



In the △ **ABC**, let the lines **P**p, **Q**q, **R**r,..., parallel to the base **BC**, divide **AB** into equal parts.

To prove that they also divide AC *into equal parts.*

Let **AX** be drawn parallel to **BC**.

Then ∵ the parallels **AX**, **P**p, **Q**q, **R**r, **T**t and **BC** make equal intercepts on the transversal **AB**, they also make equal intercepts on the transversal **AC** ;

$$\therefore\ \mathrm{A}p=pq=qr=\ldots\ldots$$

It is useful to know the lengths of the parallels in terms of the base.

The parallels $p1$, $q2$, $r3$, $t4$, divide the base **BC** into five equal parts.

Thus $\mathrm{P}p=\mathrm{B}1=\frac{1}{5}\mathrm{BC}$, $\mathrm{Q}q=\mathrm{B}2=\frac{2}{5}\mathrm{BC}$, $\mathrm{R}r=\mathrm{B}3=\frac{3}{5}\mathrm{BC}$, $\mathrm{T}t=\mathrm{B}4=\frac{4}{5}\mathrm{BC}$.

In particular, if a parallel be drawn through the middle point of **AB**, its length $=\frac{1}{2}\mathrm{BC}$.

EXERCISES XXII.

1. In the Fig. of Theorem 26, if **AB** is parallel to **CD**, and **PQ=QR** and $pq=qr$, then **EF** is parallel to **AB** or **CD**.

2. **The straight line drawn through the middle point of one side of a triangle parallel to the base bisects the other side and is half the base.** [This is a particular case of the Corollary.]

3. Obliques are drawn to a straight line from a given point without it; prove that their middle points lie on a parallel to the given line.

4. **The straight line joining the middle points of the oblique sides of a trapezium is parallel to the parallel sides and half their sum.**

5. The diagonals of a trapezium are bisected by the line joining the middle points of the oblique sides.

6. **P**, **Q** are the middle points of the opposite sides **AB**, **CD** of a parallelogram **ABCD**; prove that **BQ**, **DP** trisect the diagonal **AC**.

7. **The straight line joining the middle points of the sides of a triangle is parallel to the base and half of it.**

8. **The straight lines joining the middle points of the sides of a triangle divide it into four congruent triangles.**

9. The straight lines that join the middle points of the adjacent sides of a quadrilateral form a parallelogram, whose perimeter is equal to the sum of the diagonals of the quadrilateral.

10. Straight lines joining the middle points of opposite sides of any quadrilateral bisect each other.

11. **X is the middle point of AB; from A, X, B, perpendiculars AM, XY, BN are drawn to a given stright line PQ; prove that**
   (i) **XY=half the sum of AM, BN, if A and B are both on the same side of PQ.**
   (ii) **XY=half the difference of AM, BN, if A and B are on opposite sides of PQ.**

12. From the angular points of a parallelogram, perpendiculars are drawn to a straight line outside the parallelogram. Prove that the sum of the perpendiculars from one pair of opposite angles is equal to the sum of the perpendiculars from the other pair.

13. The parallel sides of a trapezium are 2·2 cm. and 3·8 cm. and the oblique sides are 1·5 cm. and 1·8 cm. Draw a figure of the trapezium and find the length of the line that joins the middle points of the oblique sides. Verify your answer by measurement.

14. One of the sides of a triangle is divided into 10 equal parts and parallels to the base through the points of section are drawn. Find the length of each parallel if the base is 4·7 cm. Verify by measurement.

15. Draw a plan of the triangle whose sides are 500 yd., 800 yd. and 900 yd. respectively, the scale being 1 cm. to 100 yd. Find the lengths of the straight lines joining the middle points of the sides and verify your answer by measurement.

PRACTICAL GEOMETRY

MATHEMATICAL INSTRUMENTS

In Practical Geometry, the following instruments are required.

1. A hard pencil.

2. A pair of compasses with a hard pencil point.

3. A pair of dividers.

4. A flat ruler with one edge graduated in inches and tenths of an inch, and the other in centimetres and millimetres.

5. Two set squares ; one with angles of 45°, 45°, 90°, and the other with angles of 30°, 60°, 90°.

6. A semi-circular protractor.

We shall now explain the use of these instruments.

1. **Hard Pencil.** The pencil should be sharpened into a fine chisel edge. It is used, with the help of a ruler, to draw straight lines.

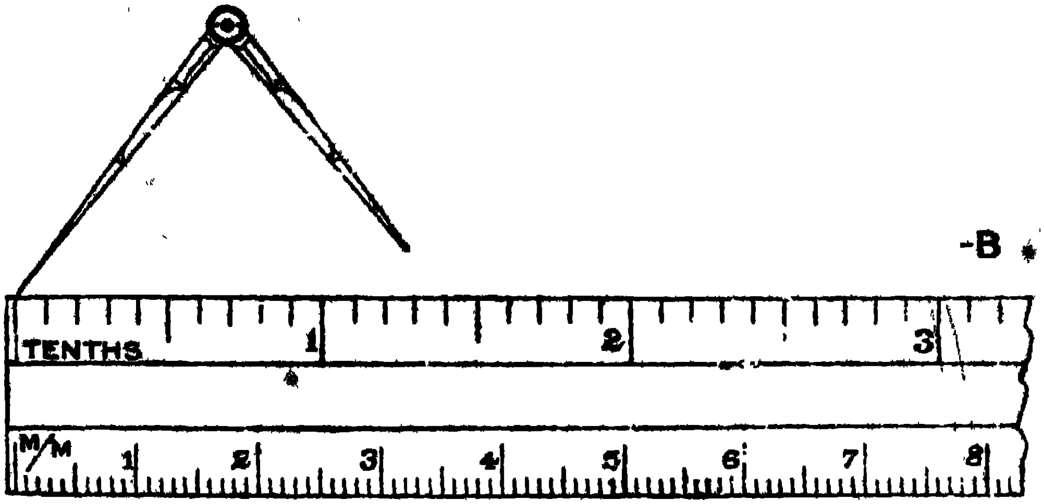
2. **The Compasses.** This instrument is used to draw circles and arcs of circles.

The compass pencil should be sharpened into a fine needle point, and must be so adjusted that the two legs be of the same length. Then use the compasses by holding them by the head only.

To draw a circle of given radius, open the compasses and apply them to the scale to see that the legs are stretched to the required length ; then draw the circle.

3. **The Dividers.** They are used to measure the distance between points, and to compare the lengths of straight lines.

To measure any distance, open the dividers and adjust the legs, so that their ends may be at the right distance; then transfer the distance to the scale.



Thus to measure the distance AB, place one end at A and see that the other end is at B. Then transfer the length to the scale by applying the dividers to the scale as shown in the diagram. In doing so, hold the dividers nearly flat and by the head only.

AB is thus seen to measure 1·4 inches.

4. **The Flat Ruler.** This is used to draw straight lines, to join points, and to produce a straight line already drawn.

It is also used for measuring lengths of straight lines. To measure a straight line, place the scale along the line, so that its graduations may be as near to the line as possible; then read off the length.

It should be noted here that measurement by the dividers is always more accurate.

Set Squares. Set squares are used

(i) to make angles of 30°, 60°, and 45°

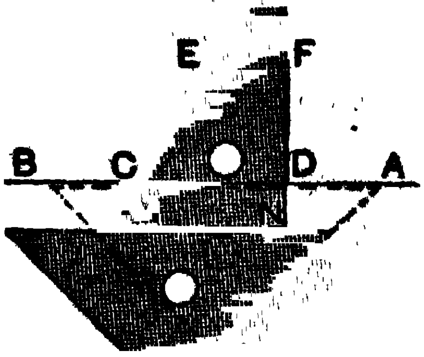
(ii) to draw perpendiculars;

and (iii) to draw parallels.

They are right-angled triangles, so that the longest side in each is the hypotenuse.

(i) To make an angle of 60°

Draw a st. line AB. Place a set square, with its hypotenuse in contact with AB; then slide it down, so that the hypotenuse is parallel o AB.



Take a set square **LMN**, having the ∠**M** = 60°; and place its side **MN** along the hypotenuse of the first set square.

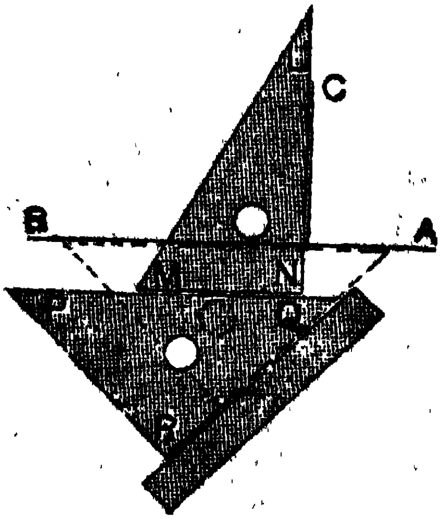
Use **ML** as a ruler to draw the line **CE**.

Then the ∠ACE = 60°.

Obs. To draw a perpendicular, use the edge **LN** as a ruler and draw **DF**. Thus **FD** is perpendicular to **AB**.

(ii) From a given point C, to draw a perpendicular to a given straight line AB.

Place a set square (represented by the dotted triangle), with its hypotenuse along AB; slide it down to the position **PQR**, so that the hypotenuse **PQ** is parallel to AB.



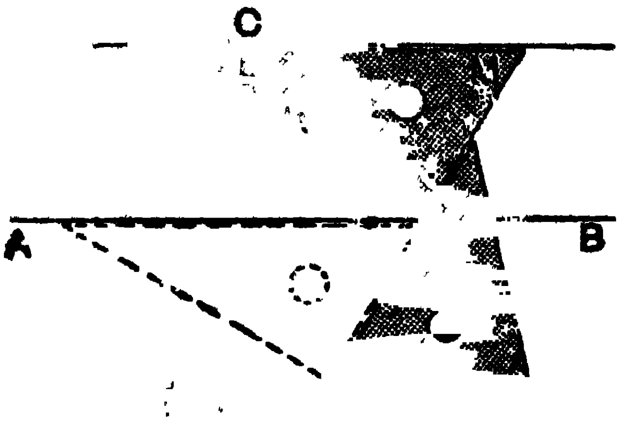
Take another set square **LMN** in which **MN** is one of the sides containing the right angle. Place **LMN** with **MN** in contact with **PQ**, and slide it along **PQ**; until the edge **LN** is just clear of the given point C.

Then use **LN** as a ruler, and draw a line through **C**.

The line so drawn is the perpendicular required.

Obs. The method is the same whether **C** is in **AB**, or without **AB**.

(iii) Through a given point C, to draw a straight line parallel to a given straight line AB.



Place a set square (represented by the dotted triangle) with its hypotenuse in contact with AB.

Place a second set square with its hypotenuse along a side of the first set square.

Hold the second set square firmly, and slide the first along the hypotenuse into the position **LMN**, so that the hypotenuse **LM** is just clear of the given point C.

Then use the edge **LM** as a ruler, and draw a line through C.

The line thus drawn is the parallel required.

Note. To draw *any two parallel straight lines*, take a set square **LMN**, and place another set square (or a ruler) in contact with **MN**; and rule a line along the edge **LM**. Then holding the second set square (or the ruler) firmly, slide **LMN** up or down, and rule a line along the same edge **LM**.

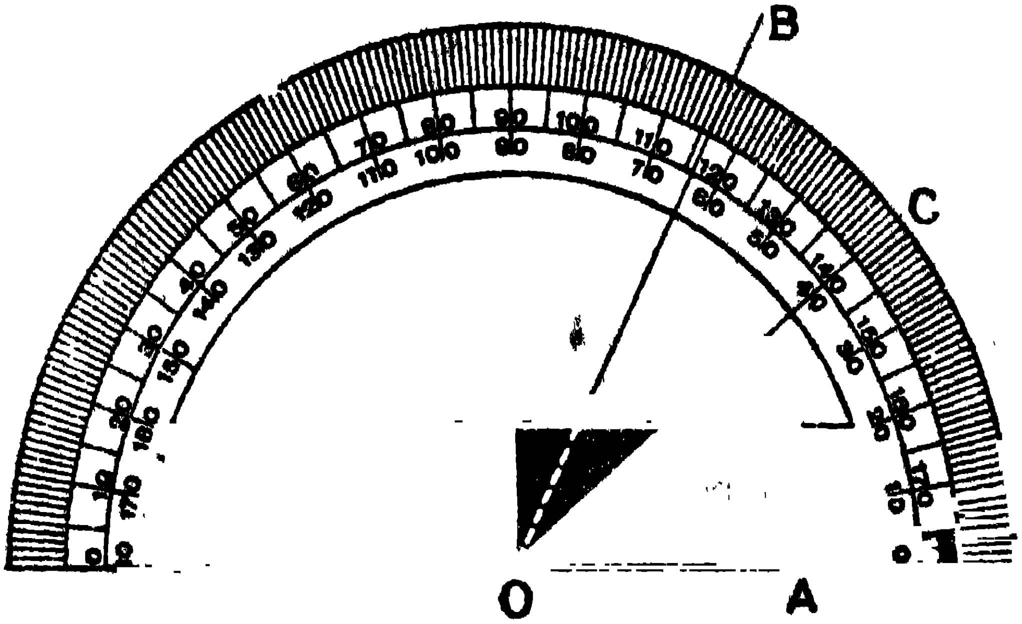
The two straight lines are parallel to one another.

6. **Semi-circular Protractor.** This instrument is used (i) to measure a given angle, (ii) to make an angle of given magnitude.

(i) To measure a given angle.

Place the protractor so that its centre (that is, the middle point of the straight edge) is at the vertex of the given angle and the edge itself along an arm. Then read off the number of degrees by observing the graduation on the circular edge with which the other arm coincides.

Thus the angle AOB, whose vertex is O, measures 64°.



(ii) To make an angle of given magnitude.

Suppose an angle of 40° is to be made at O in the st. line AO.

Place the protractor with its centre at O and the straight edge along OA. Then make a fine pencil mark at C, C being just at the 40° graduation. Now remove the protractor and join OC.

Thus AOC is an angle of 40°.

CONSTRUCTIONS

The use of Hypothetical Constructions is now permitted in the proof of Theorems ; hence the Problems that follow do not form part of a logical course, but are given only for their practical value.

In the following constructions, use will be made of a **straight ruler** and a **pair of compasses** *only* ; the ruler being used only to join points and produce straight lines ; and the compasses to draw circles with given centre and radius. It should be noted here that in all purely geometrical constructions, the use of these two instruments only and of no others, is permitted.

The accuracy of these constructions will be *theoretically proved* by means of the Theorems already established. But the student should always test the correctness of each drawing by measurement ; in doing this he may use *any instrument* he likes.

In the diagrams of the following constructions,

(i) given lines are of *medium thickness*,

(ii) required lines are *thick*,

(iii) construction lines are *fine*.

and (iv) lines required only for the proof are *dotted*.

Also, the *necessary arcs* of construction circles are only shewn (and not the whole circles), though "circles" are spoken of.

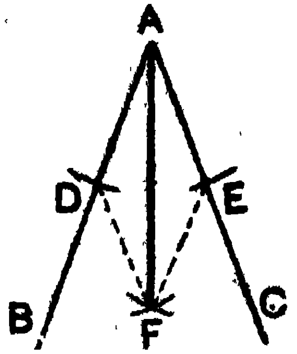
In making geometrical constructions, great care should be bestowed on the *accuracy of drawing*. For this purpose, the student is advised to construct on a large scale, for a slight inaccuracy is overlooked in a large figure than in a small one.

When required to construct a figure, the student should always *explain the construction in words*, taking care to *shew clearly all construction lines in his diagram*.

Problem 1. To bisect a given angle.

Let BAC be the given angle to be bisected.

Construction. With centre A and any radius, draw a circle cutting AB in D and AC in E.



With centres D and E and any convenient radius, draw two arcs cutting at F.

Join AF.

Then AF bisects the ∠ BAC.

Proof. Join FD, FE.

In the $\triangle^s$ ADF, AEF,

(i) AD = AE, [*Const.*

(ii) DF = EF, (radii of equal circles),

and (iii) AF is common ;

∴ the triangles are congruent ; [I. 7.

∴ the ∠ DAF = the ∠ EAF,

that is, AF bisects the ∠ BAC.

"*Convenient radius*"—a radius that enables us to make the necessary construction. In this case the radius must be such that the two construction circles may intersect.

EXERCISES XXIII.

1. Is *any* point in **AF** equally distant from **AB**, **AC** ?

2. Give a proof of this Problem, when the two construction circles intersect at a point nearer **A**. In what case would the construction fail ?

3. By folding about **AF**, shew that **AF** is an axis of symmetry of the figure **ADFE**. Hence prove that **DE** is bisected by **AF**.

4. In what case would **DE** be also an axis of symmetry of the figure **ADFE** ?

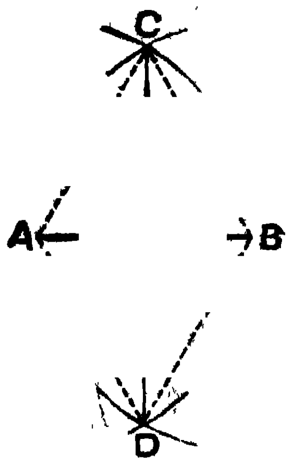
5. Take several angles and bisect them. Test the accuracy of your results by measurements.

6 Divide a given angle into 4, 8, 16,......equal parts.

7. Shew that the angle **BAC** can be bisected by the following construction :—Take any point **P** in **AB**, draw **PQ** parallel to **AC** and make **PQ**=**AP** ; then **AQ** bisects the ∠**BAC**.

8. Cut a triangle out of paper ; shew, by folding, that the three bisectors of its angles intersect at a point.

Problem 2, To bisect a given finite straight line.



Let AB be the given st. line to be bisected.

Construction. With centres A and B and any covenient radius, draw arcs of two equal circles intersecting at C and D.

Join CD cutting AB at O.

Then AB is bisected at O.

Proof. Join AC, AD, BC, BD.

In the $\triangle^s$ ACD, BCD,

(i) AC = BC, [*Const.*

(ii) AD = BD, [*Const.*

and (iii) CD is common ;

$\therefore$ the $\angle$ ACD = the $\angle$ BCD, [I. 7.

Again in the $\triangle^s$ ACO, BCO,

(i) AC = BC,

(ii) CO is common,

and (iii) the included $\angle$ ACO = the included $\angle$ BCO ;

$\therefore$ AO = BO, [I. 4.

that is, AB is bisected at O.

Note. Since the $\triangle$s **ACO, BCO** are congruent, the angles at **O** are right angles ; hence **CO** *bisects* **AB** *perpendicularly.*

EXERCISES XXIV.

1. By folding about **AB** and **CD** by turns, shew that **AB** and **CD** are each an axis of symmetry of the figure **ACBD** and that the angles at **O** are all right angles. Hence prove that the diagonals of a rhombus bisect each other at right angles, and also the angles through which they pass.

2. Why is any point in **AB** equidistant from **C** and **D**, and any point in **CD** from **A** and **B** ?

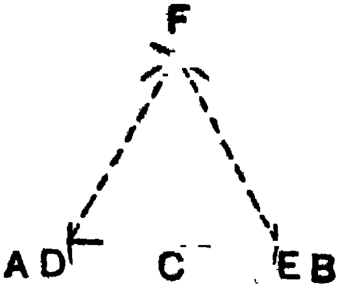
3. Divide a given straight line into 4, 8, 16,.... equal parts.

4. Produce a st. line **AB** to **C**, so that **AC** may be 3 times **BC**.

5. Prove that any straight line **AB** can be bisected by the following construction :—Draw **AC** making any angle with **AB**, produce **AC** to **D** making **CD** = **AC**, join **BD**, and draw **CE** parallel to **BD** ; then **CE** bisects **AB**.

Problem 3. To draw a straight line perpendicular to a given straight line from a given point in it.

Let AB be the given st. line and C the given pt. in it, from which a perp. is to be drawn.



First Method

Construction. With centre C and any convenient radius, draw a circle cutting AB in D and E.

With centres D and E, and any radius greater than CD, draw two arcs intersecting at F.

Join CF.

Then CF is perpendicular to AB.

Proof. Join DF, EF.

In the $\triangle^s$ FCD, FCE,

(i) CD = CE, [*Const.*

(ii FC is common,

and (iii) DF = EF, (radii of equal circles) ;

the ∠FCD = the ∠FCE. [I. 7.

But these being adjacent angles are rt. angles ;

∴ CF is perpendicular to AB.

Second Method

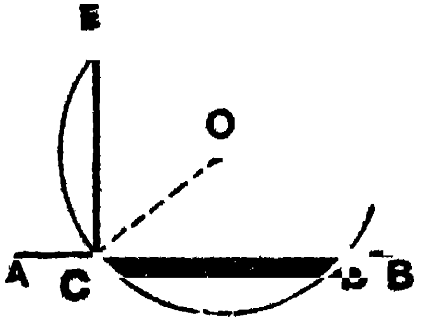
Construction. Take any convenient pt. O outside AB.

With centre O and radius OC, draw a circle cutting AB in D.

Join DO, and produce it to cut the circle in E.

Join CE.

Then CE is perpendicular to AB.



Proof. Join OC.

Then ∵ OC = OD ; ∴ the ∠ OCD = the ∠ODC ;

and ∵ OC = OE, ∴ the ∠ OCE = the ∠ OEC ;

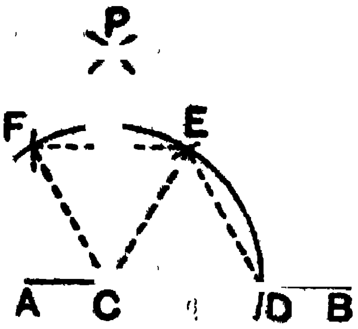
∴ the whole ∠ DCE = the ∠ CDE + the ∠CED ;

∴ the ∠ DCE is a right angle. [I. 16, *Cor.* 3.

∴ CE is perpendicular to AB.

THIRD METHOD

Construction. With centre **C** and any radius, draw an arc **DEF** cutting **AB** in **D**.



With centre **D** and the same radius, draw an arc cutting the arc **DEF** in **E**.

With centre **E** and the same radius, draw an arc cutting the arc **DEF** in **F**.

With centres **E** and **F** and any radius, draw two arcs to intersect at **P**.

Join **CP**.

Then **CP** is perpendicular to **AB**.

Proof. Join **DE**, **EF**, **CE**, **CF**.

Now **CD**, **CE**, **DE**, **EF** and **CF** are all equal, being radii of equal circles ; [*Const.*

∴ **CDE** and **CEF** are equilateral triangles ;

so that the ∠ **DCE** = the ∠ **ECF** = 60°.

And by construction **CP** bisects the ∠ **ECF** ;

∴ the ∠ **ECP** = 30°.

∴ the ∠ **DCP** = the ∠ **DCE** + the ∠ **ECP** = 90° ;

∴ **CP** is perpendicular to **AB**.

Obs. If the given point coincides with either end of the given line **AB**, then the construction for drawing the perpendicular *without producing* **AB** is the same as given here.

EXERCISES XXV.

1. At a given point in a given straight line, make an angle equal to half a right angle.

2. Construct a right-angled isosceles triangle.

3. In the base **BC** of a triangle **ABC**, any point **D** is taken ; draw a straight line such that, if the triangle **ABC** be folded about this line, the point **A** shall be on the point **D**.

4. Cut out of paper an isosceles triangle; shew, by folding, that the perpendicular at the middle point of the base passes through the vertex and bisects the vertical angle.

5. Cut out of paper an acute-angled triangle; shew, by folding, that the perpendiculars from the middle points of the sides meet at a point.

6. Shew that a perpendicular to **AB** can be drawn from **A**, without producing **AB**, by the following construction;—With centres **A** and **B** and any convenient radius, draw two circles intersecting at **C**; join **AC**, **BC**; and produce **BC** to **D** making **CD** = **CB**; join **AD**; then **AD** is perpendicular to **AB**.

Problem 4. To draw a straight line perpendicular to a given straight line from a given point without it.

Let AB be the given st. line and C the given pt. without it, from which a perp. is to be drawn to AB.

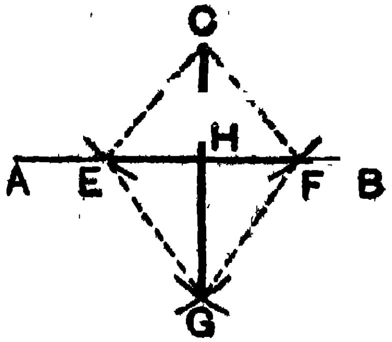
First Method

Construction. With centre C and any convenient radius, draw an arc so as to cut AB at E and F.

With centres E and F and any convenient radius, draw two arcs intersecting at G.

Join CG, cutting AB in H.

Then CH is perp. to AB.



Proof. Join EC, EG, FC, FG.

In the Δ* ECG, FCG,

(i) EC = FC, [*Const.*

(ii) EG = FG, (radii of equal circles),

and (iii) CG is common;

∴ the ∠ECG = the ∠FCG. [I. 7.

Again in the Δ* CHE, CHF,

(i) CE = CF,

(ii) CH is common,

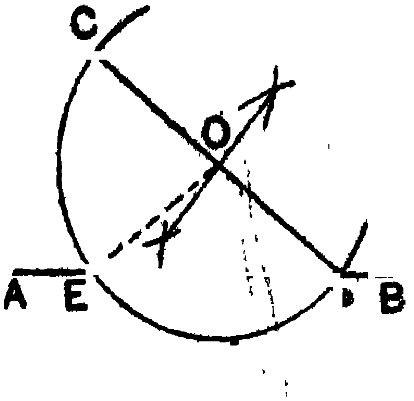
and (iii) the included ∠ECH = the included ∠FCH;

∴ the ∠CHE = the ∠CHF, [I. 4.

But these being adjacent angles are rt. angles ;

∴ CH is perpendicular to AB.

SECOND METHOD



Construction. Take any convenient pt D in AB. Join CD and bisect it at O.

With centre O and radius OD or OC, draw an arc cutting AB in E.

Join CE.

Then **CE** is perpendicular to AB.

Proof. Join OE.

Then ∵ OE = OD ; ∴ the ∠OED = the ∠ODE ;

and ∵ OE = OC ; ∴ the ∠OEC = the ∠OCE ;

∴ the whole ∠DEC = the ∠EDC + the ∠ECD ;

∴ the ∠DEC is a right angle. [I. 16. *Cor.* 3.

∴ CE is perpendicular to AB.

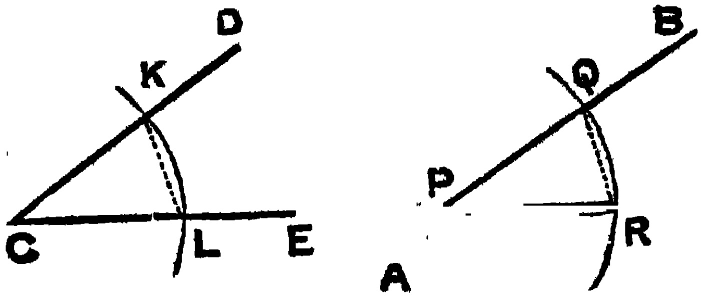
EXERCISES XXVI.

1. Prove that a perpendicular to **AB** from **C** can be drawn by the following construction :—With centre **C** and any convenient radius, draw a circle cutting **AB** in **E**, **F** ; bisect **EF** at **D** ; join **CD** ; then **CD** is perpendicular to **AB**.

2. Cut out of paper an acute-angled triangle ; shew by folding that the perpendiculars from its three angles to the opposite sides meet at a point.

3. Through a given point, draw a straight line such that the perpendiculars drawn to it from two other points may be on opposite sides of it and equal to one another.

Problem 5. At a given point in a given straight line, to make an angle equal to a given angle.



Let AB be the given st. line and P the given point in it; and let DCE be the given angle.

To make an angle at P equal to the ∠DCE.

Construction. With centre C and any radius, draw an arc cutting CD in K and CE in L.

With centre P and the same radius, draw an arc QR cutting AB in Q.

With centre Q and radius equal to KL, draw an arc cutting the arc QR in R.

Join PR.

Then QPR is the angle required.

Proof. Join KL, QR.

Then in the △ˢ QPR, KCL,

(i) QP = KC,
(ii) PR = CL,
and (iii) QR = KL; [*Const.*

∴ the ∠QPR = the ∠KCL or the ∠DCE. [I. 7.

Obs. The same construction gives *another angle* equal to the ∠DCE on the other side of AB.

EXERCISES XXVII.

1. At a given point in a given line, make an angle equal to the supplement to a given angle.

2. Make an angle (i) double of a given angle and (ii) half of a given angle.

3. **AB** is a given straight line and **P** a point without it; through **P**, draw a straight line parallel to **AB**.

4. Having given two points, one of which is in a given line, find another point in the given line, such that the sum of its distances from the two given points is given.

5. In Ex. 4, find the point when the difference of the distances is given.

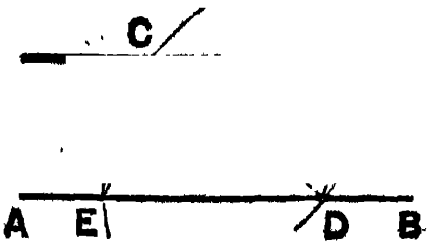
6. Construct a triangle, having given the base, one of the angles at the base and the sum of the sides.

7. In Ex. 6, construct the triangle, when the difference of the sides is given.

Problem 6. Through a given point, to draw a straight line parallel to a given straight line.

Let AB be the given st. line, and C the given pt. through which a st. line is to be drawn parallel to AB.

FIRST METHOD



Construction. In AB, take any pt. D.

With centre D and radius DC, draw an arc cutting AB at E.

With centre C and radius CD, draw an arc DF.

With centre D and radius CE, draw an arc cutting the arc DF at F, F being on the same side of AB as C.

Join CF.

Then CF is parallel to AB.

Proof. Join CE, CD and DF.

In the Δ^s CDE, DCF,

(i) DE = CF, (ii) CE = DF,

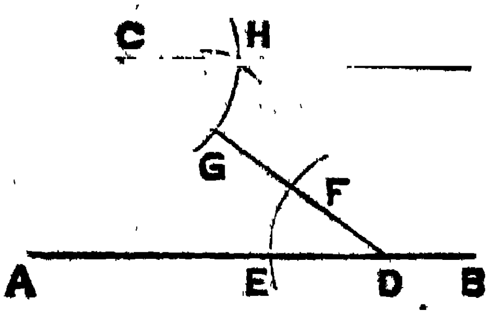
and (iii) CD is common;

∴ the ∠CDE = the ∠DCF; [I. 7.

and these are alternate angles,

∴ CF is parallel to AB. [I. 14.

SECOND METHOD



Construction. In **AB**, take any point **D**.

Join **CD**.

With centre **D** and any convenient radius, draw an arc **EF** cutting **AB** at **E** and **CD** at **F**.

With centre **C** and the same radius, draw an arc **GH** cutting **CD** at **G**.

With centre **C** and radius equal to **EF**, draw an arc cutting the arc **GH** at **H**, so that **H** may be on the side of **CD** remote from **AB**.

Join **CH**.

Then **CH** is parallel to **AB**.

Proof. The $\triangle^s$ **EDF**, **HCG** are congruent,

so that the ∠**EDF** = the ∠**HCG**. And these are alt. angles ;

∴ **AB** and **CH** are parallel.

These two methods are substantially the same ; but this is more practical.

EXERCISES XXVIII.

1. Construct a square on a given straight line.

2. Construct a rectangle, having given two adjacent sides.

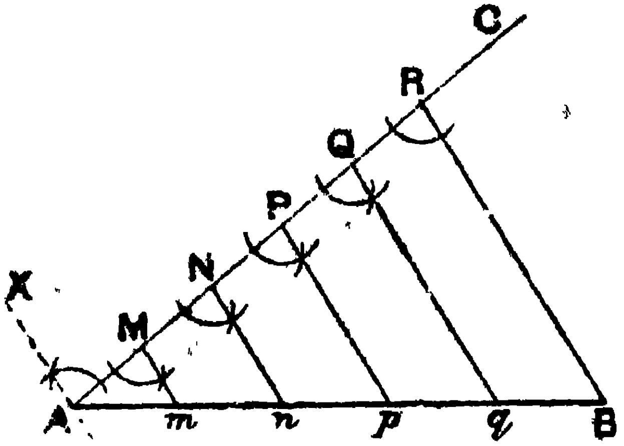
3. Construct a parallelogram, having given two adjacent sides and the angle included between them.

4. Prove the following construction for trisecting a given straight line **AB** :—With centres **A** and **B** and radius **AB**, draw two circles intersecting at **C** ; join **AC**, **BC** ; bisect the ∠s **ABC**, **BAC** by lines **BD**, **AD** meeting at **D** ; draw **DE**, **DF** parallel to **AC**, **BC**, meeting **AB** in **E**, **F** ; then **AB** is trisected at **E** and **F**.

5. From a given point, draw a straight line making with a given straight line an angle equal to a given angle. How many solutions are there ?

6. **AB**, **AC** are two straight lines ; from a point **P** draw a straight line **PQ** to **AB**, so that **PQ** may be bisected by **AC**.

Problem 7. To divide a given straight line into any number of equal parts.



Let AB be the given st. line, which is to be divided into any number of equal parts (say *five*).

First Method

Construction. Through A, draw a st. line AC, making any angle with AB.

Along AC, mark off five equal lengths AM, MN, NP, PQ, QR.

Join RB; and through M, N, P, Q, draw parls. Mm, Nn, Pp, Qq, to RB cutting AB at m, n, p and q.

Then AB is divided as required at the pts. m, n, p, q.

Proof. Through A, draw AX parallel to RB.

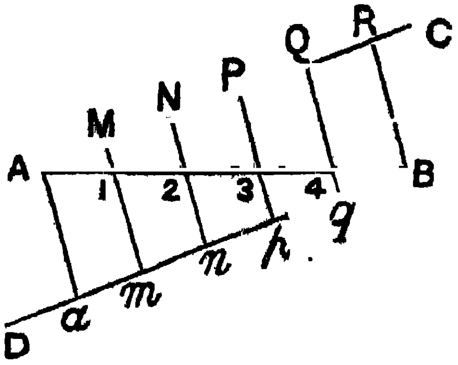
Then AX, Mm, Nn, Pp, Qq, and RB are parallel to one another. And these parallels make equal intercepts AM, MN, NP, PQ, QR, on AC;

$\therefore$ they also make equal intercepts on AB. [I. 26.

$\therefore$ Am = mn = np = pq = qB.

Thus AB is divided into five equal parts.

SECOND METHOD



Construction. Through A, draw a st. line AC making any angle with AB.

Along AC, mark off five equal lengths AM, MN, NP, PQ, QR.

Through B, draw BD ∥ to AC, and mark off along BD, lengths B*q*, *qp*, *pn*, *nm* and *ma*, each equal to AM.

Join RB, Q*q*, P*p*, N*n*, M*m* and A*a*.

Then AB is divided into five equal parts at the pts. marked 1, 2, 3, 4.

[Prove by I. 23 and 26.]

EXERCISES XXIX.

1. Trisect a given straight line. [See also (xxviii), 4].

2. Divide a straight line into 7 equal parts.

3. Divide a straight line of 10·8 cm. into 2 parts, so that one part may be $\frac{2}{7}$ of the other.

4. Divide a straight line of 11·2 cm. in the ratio of 3 : 5.

5. Draw a straight line equal to (i) $1\frac{1}{4}$ and (ii) 2·3, of a given line.

TRIANGLES

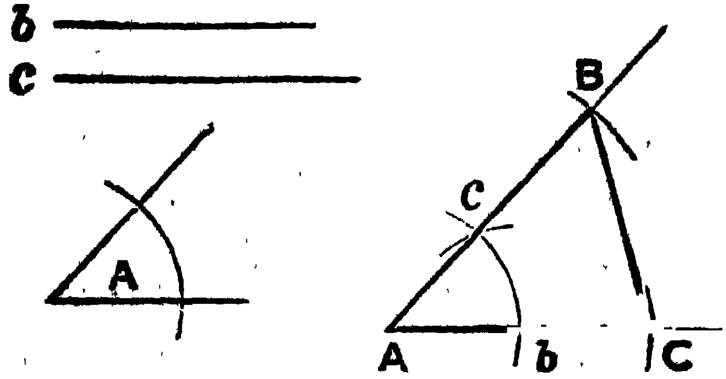
Notation. It is usual to denote the *angles* of a triangle ABC by the capital letters A, B, C and the sides opposite to them by the small letters *a*, *b*, *c* respectively. Thus the six parts of a triangle ABC are A, B, C and *a*, *b*, *c*.

In subsequent Problems, we shall make use of this notation.

We have seen that two triangles are equal in all respects when three parts of the one are equal to the corresponding three parts of the other. We have also pointed out the two cases in which this does not happen. [See p. 69.]

Hence we generally conclude that the *size* and *shape* of a triangle can be determined, if out of its six parts, three parts *independent* of one another are given. In other words *three independent data are necessary to construct a triangle.*

Problem 8. To construct a triangle having given two sides and the included angle (*b*, c, A).



Let *b*, *c* be the two sides and A the included angle.

Construction. Make the ∠CAB = the given ∠ A.

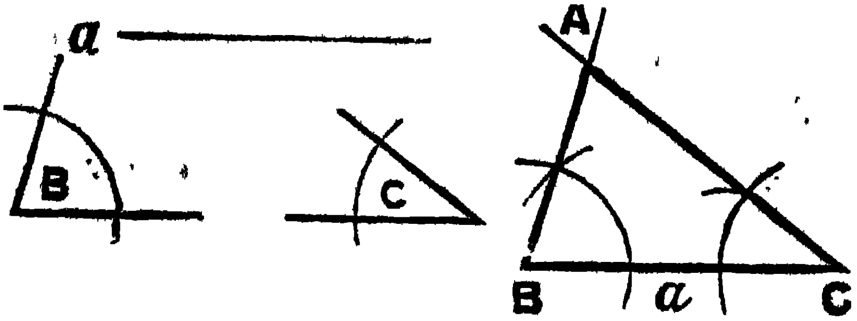
With centre A and radii *c* and *b*, draw two arcs cutting AB, AC in B and C respectively.

Join BC.

Then ABC is the required triangle.

Proof. By construction, AC = *b*, AB = *c* and the ∠CAB = A.

Problem 9. To construct a triangle having given two angles and a side (a, B, C).



CASE (i). When B, C and a are given.

Construction. Take any st. line BC equal to a.

On the same side of BC, make the $\angle CBA = B$ and the $\angle BCA = C$.

Let BA, CA intersect at the pt. A.

Then ABC is the triangle required.

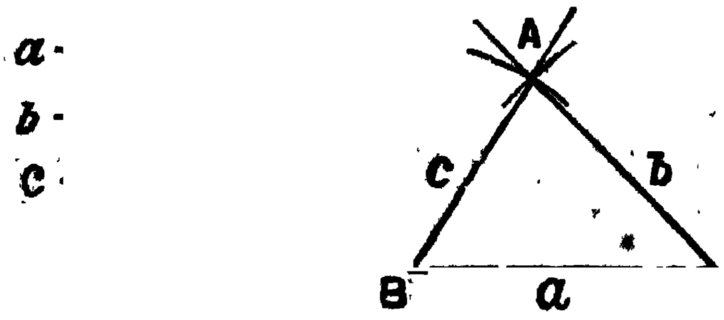
Proof. By construction, the $\angle ABC = B$, the $\angle ACB = C$ and the side $BC = a$.

CASE (ii). When A, B and a are given.

Since $C = 180^\circ - (A + B)$, ∴ C is known.

Then proceed as before.

Problem 10. To construct a triangle having its sides equal to three given straight lines (a, b, c).



Let a, b, c be the three given straight lines.

Construction. Draw any st. line BC equal to a.

With centre B and radius c, draw an arc of a circle.

With centre C and radius b, draw an arc, intersecting the former arc at A.

Join AB, AC.

Then ABC is the triangle required.

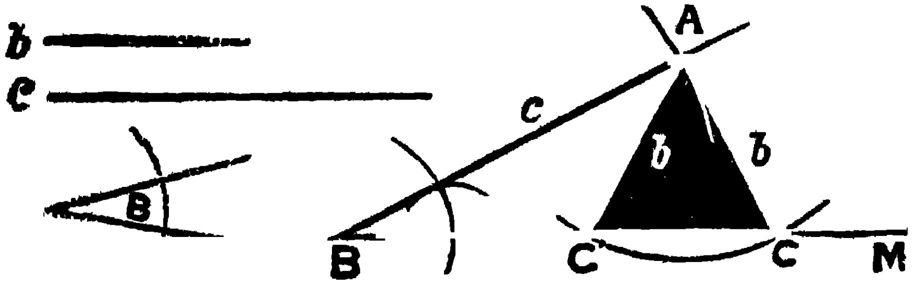
Proof. By construction, $BC = a$, $CA = b$, $AB = c$.

Note 1. It should be observed that the construction is possible only, *when any two of the given lines, a, b, c are together greater than the third*; for otherwise the arcs of construction circles may not cut one another.

Note 2. The construction circles would cut again on the other side of BC. Thus we have *two* triangles one on each side of BC.

Shew how to construct this triangle.

Problem. 11. To construct a triangle having given two sides and an angle opposite to one of them (b, c, B).



Let b, c and B be given.

Construction. Take any st. line BM.

At B, make the $\angle MBA = B$.

With centre B and radius c, draw an arc of a circle, cutting AB at A.

With centre A and radius b, draw an arc of a circle, and let it cut BM at C, C′, both these pts. being on the same side of B.

Join AC, AC′.

Then the *two* Δ^s ABC, ABC′ are the triangles required.

Proof. By construction, $AB = c$, $AC = AC' = b$,

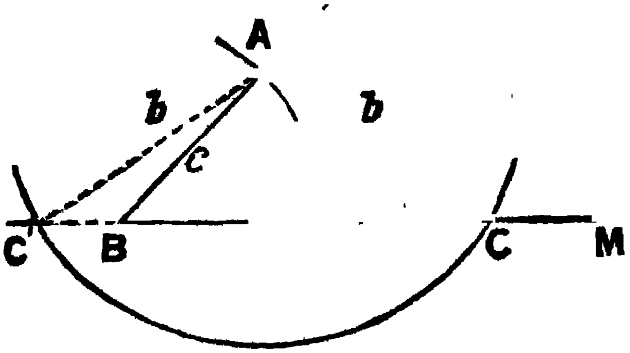
and the $\angle ABC$ or the $\angle ABC' = B$.

Note.

Since *two* triangles can be constructed each having the given parts, this solution is known as the **ambiguous case,** and will occur **when b is less than c but greater than the perpendicular from A to BM.**

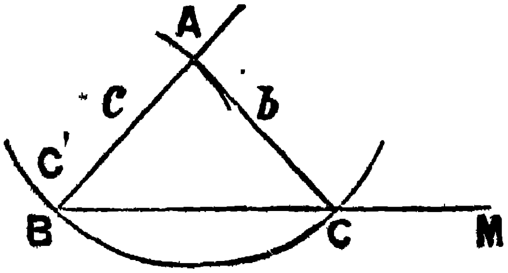
The student will do well to notice the following 4 cases of solution.

(i) If $b > c$, then the pt. **B** is within the circle whose centre is **A**, and therefore **C, C′** are on opposite sides of **B**.



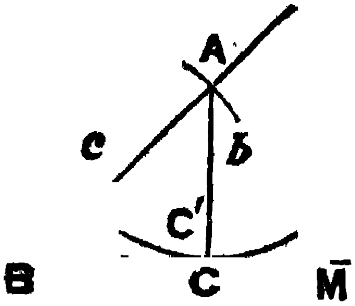
Thus here is only **one** triangle, *viz.*, **ABC**, having the given parts.

(ii) If $b = c$, the circle passes through the pt. **B**; thus **C′** coincides with **B**.



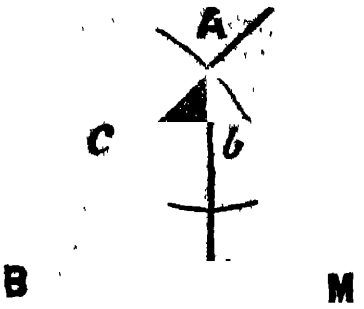
Hence here is only **one** triangle, *viz.*, **ABC**, satisfying the given conditions.

(iii) If $b =$ the perpendicular from **A** to **BM**, the circle meets **BM** in *one* point only, *viz.*, **C**, so that **C′** coincides with **C**.



Thus in this case also there is only **one** triangle, *viz*, **ABC**, which is the required triangle.

(iv) If b is less than the perpendicular from **A** to **BM**, the circle cannot cut **BM**; and therefore in this case **no triangle** can be constructed.

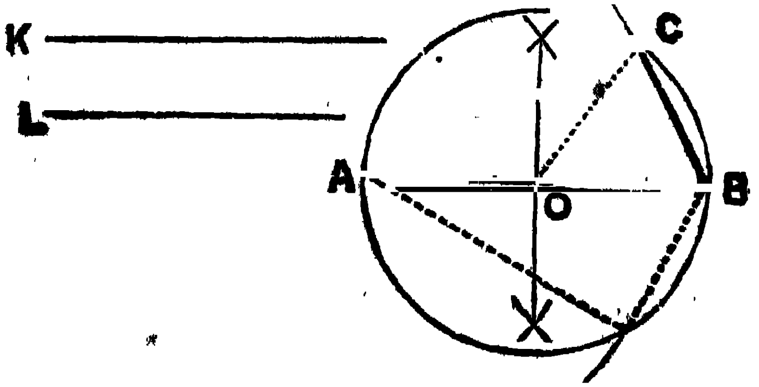


Problem 12. To construct a right-angled triangle, having given the hypotenuse and one side.

Let K be the hypotenuse and L the given side.

First Method

Construction. Take the st. line AB equal to K. Bisect AB at O. With centre O and radius OA or OB, draw a circle. With centre A and radius equal to L, draw a circle cutting the first circle at C. Join AC, BC.



Then ABC is the required triangle.

Proof. Join OC.

Then because OA = OC = OB;

∴ the ∠OAC = the ∠OCA;

and the ∠OBC = the ∠OCB;

∴ the ∠OAC + the ∠OBC = the ∠OCA + the ∠OCB

= the ∠ACB.

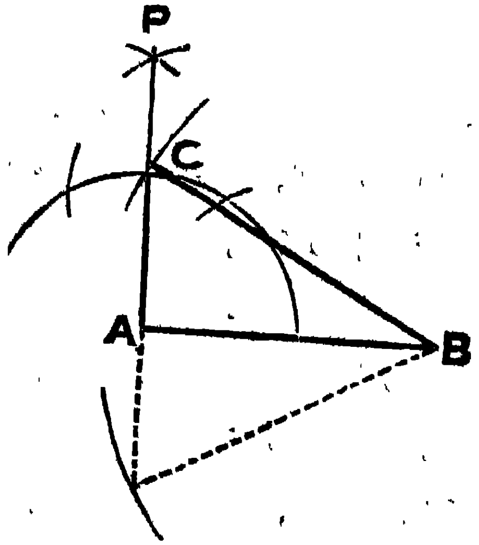
∴ the ∠ACB is a right angle. [I. 16. *Cor.* 3.

Thus in the △ABC, AB = K, AC = L
and the ∠ACB = a rt. angle.

Note. Since the second circle cuts the first circle at another point, the Problem has another solution, indicated by the dotted lines below.

SECOND METHOD

Construction. Take the st. line AB equal to L.



Draw AP at right angles to AB.

With centre B and radius equal to K, draw an arc cutting AP at C.

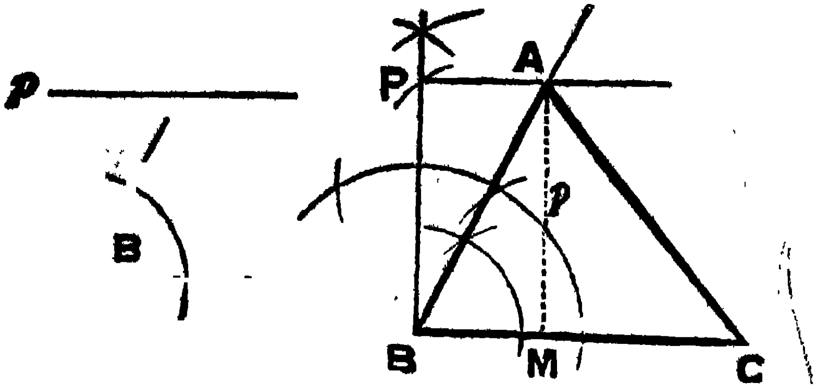
Join BC.

Then ABC is the required triangle.

Proof. By construction, AB = L, BC = K and the angle at A is a right angle.

Note. Since the construction circle cuts PA produced at another point, there is another solution indicated by the dotted lines.

Problem 13. To construct a triangle, havi[illegible]iven the base, an angle at the base and the perpen[illegible] from the vertex to the base.



Let BC be the given base, B the given angle, and p the perpendicular from the vertex to BC.

Construction. At B in BC, make the ∠CBA = B.

Draw BP perp. to BC, making BP = p.

Through P, draw PA ∥ to BC, cutting BA in A.

Join AC.

Then ABC is the triangle required.

Proof. Draw AM perp. to BC.

Since BPAM is a parm., AM = PB = p.

And the ∠CBA = B and BC is the given base.

Def. The perpendicular from a vertex to the opposite side considered as base, is called an **altitude** or **height** of a triangle.

Thus a triangle has *three* altitudes.

EXERCISES XXX.

[*Graphical.*]

1. Construct a right-angled triangle, having given the hypotenuse $=6\cdot63$ cm. and a side $b=5\cdot85$ cm. Measure the side a and find $\sqrt{c^2-b^2}$. Compare these two values.

2. Construct a triangle, whose sides are respectively 1·34 in., 1·25 in. and 1·3 in.

By drawing the three medians of this triangle, shew that they are concurrent. Measure the two parts of each median. What is the ratio between them ?

3. Construct a triangle **ABC**, having given $a=3·6$ cm., $b=4·2$ cm. and $c=5·8$ cm.

Bisect the angle **C** by **CO**, cutting **AB** in **O**. Measure **AO**, **BO** and find to 2 places of decimals the value of $\frac{\mathbf{BO}}{\mathbf{AO}}$. Compare this value with that of $\frac{a}{b}$.

4. Construct a triangle **ABC**, having given $a=6·2$ cm., $b=5·2$ cm. and **B** $=40°$.

Shew that there are *two* solutions. Measure the two values of c and shew that the two values of **A** are supplementary.

5. Construct a triangle **ABC**, having given $a=5·3$ cm., $b=8·5$ cm. and **B** $=54°$.

Shew that, unlike the ast Example, here is only *one* solution. Why ? By measurement, find c.

6. Construct a right-angled triangle, having given the hypotenuse $=3·75$ cm. and **A** $=34°$.

Draw **CD** perpendicular to c. Measure **AD**, **BD** and **CD**. Find the product of **AD** and **BD**, and also the square of **CD**. Compare the two results.

7. Construct an isosceles triangle, having given the perpendicular from the vertex to the base $=12·12$ cm. and each equal side $=14$ cm.

Measure the base. Is the triangle equilateral ?

[*Theoretical.*]

8. Construct an equilateral triangle on a given base.

9. Construct an isosceles triangle on a given base, having each of the sides equal to a given straight line.

10 Construct a right-angled triangle, having given the hypotenuse and the perpendicular on it from the right angle.

[*Hint.* Draw a semi-circle on the hypotenuse. Use (XIV), 19.]

11. Construct a right-angled triangle having given the base and an angle at the base.

12. Construct a right-angled triangle, having given one side and the opposite angle.

13. Construct an equilateral triangle of given perimeter.

14. Construct an isosceles triangle, having given the base and the sum of its sides.

15. Construct a triangle, having given the base, the altitude and a side.

16. Construct an isosceles triangle, having given the base and th sum of the altitude and one of the equal sides.

17. Construct an isosceles triangle, having given the vertical angle and the altitude.

18. Construct an isosceles triangle, having given the altitude and the perimeter.

19. Construct a triangle, having given the perimeter and the angles at the base.

20. Construct a triangle, having given the base, a side and the median that bisects the base.

21. Construct a triangle, having given the base, the altitude and the median that bisects the base.

22. Construct a right-angled triangle, having given the hypotenuse and sum of the sides.

23. Construct an isosceles triangle, having given the base and the sum of the vertical angle and an angle at the base.

24. Construct a triangle of given perimeter, having its angles equal to those of a given triangle.

25. Construct a right-angled triangle, having given the perimeter and an angle.

26. Construct a triangle, having two angles and the length of the perpendicular from the third angle to the opposite side.

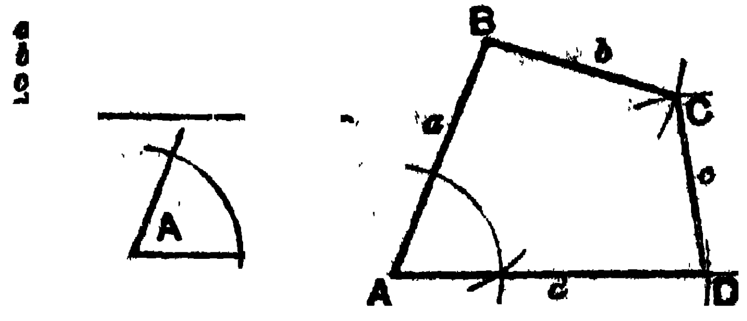
27. Construct a right-angled triangle, having given the base and the perpendicular from the right angle on the hypotenuse.

When would the construction fail ?

QUADRILATERALS

We have seen that *three* independent data are necessary to construct a triangle. The student will presently see that *five* independent data are required to construct a quadrilateral.

Problem 14. To construct a quadrilateral, having its sides equal to four given straight lines and an angle equal to a given angle.



Let a, b, c, d, be the four given st. lines and A the angle between the sides equal to a and d.

Construction. Take a st. line AB equal to a.

Make the $\angle$ BAD = A and from AD cut off AD = d.

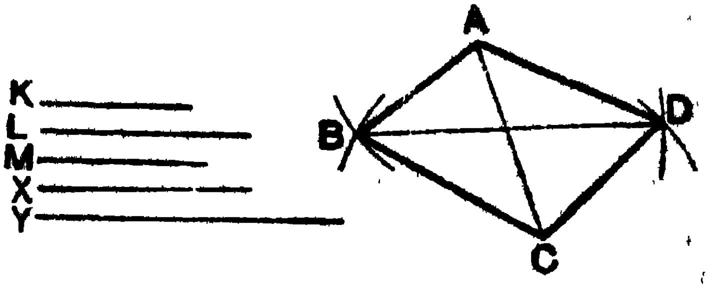
With centres B and D, and radii b and c respectively, draw two arcs intersecting at the point C.

Join BC, DC.

Then ABCD is the quadrilateral required.

Proof. By construction, the sides are equal to a, b, c, d and the $\angle$BAD = A.

Problem 15. To construct a quadrilateral, having given three sides and the diagonals.



Let K, L, M be the given sides and X, Y the given diagonals.

Construction. Take a st. line AC equal to X.

With centres A and C and radii K and L respectively, draw two arcs intersecting at B.

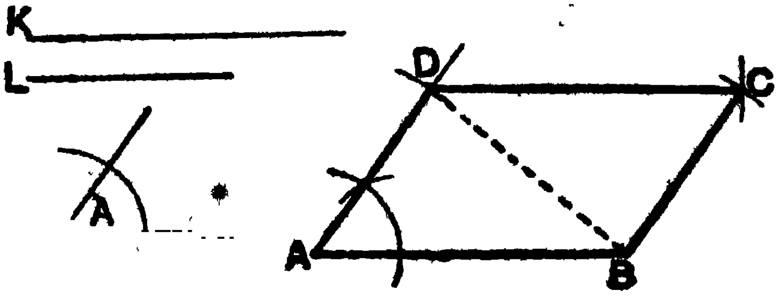
With centres B and C and radii Y and M respectively, draw two arcs intersecting at D.

Join AB, BC, CD, BD and AD.

Then ABCD is the required quadrilateral.

Proof. By construction, AB=K, BC=L, CD=M, AC=X and BD=Y.

Problem 16. To construct a parallelogram, having given two adjacent sides and the included angle.



Let K and L be the given sides, and A the given angle.

First Method

Construction. Take a st. line AB equal to K.

At A, make the ∠ BAD equal to A.

With centre A and radius L, draw an arc cutting AD in D.

With centres D and B and radii K and L, draw two arcs intersecting at C.

Join DC, BC.

Then ABCD is the parallelogram required.

Proof. Join BD.

Then in the $\triangle^s$ ABD, CDB,

(i) AB = CD, (ii) AD = CB, [*Const.*

and (iii) BD is common,

∴ the triangles are congruent ;

so that the $\angle$ ABD = the $\angle$ CDB,

and the $\angle$ ADB = the $\angle$ CBD,

And these are pairs of alternate angles ;

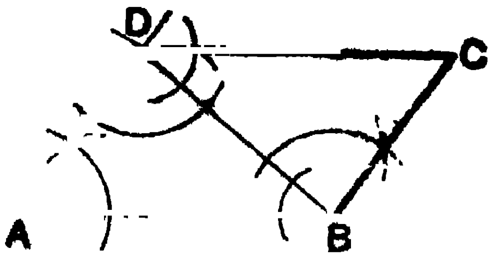
∴ AB is parallel to CD, and AD to CB,

that is, ABCD is a parallelogram ;

and in it, AB = K, AD = L, and the $\angle$ BAD = A.

SECOND METHOD

Construction. Take a st. line AB equal to K.



At A, make the $\angle$ BAD = A.

With centre A and radius L, draw an arc cutting AD in D.

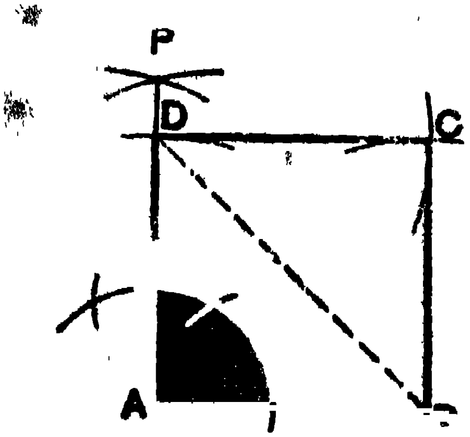
Through D, draw DC parallel to AB,

and through B, draw BC parallel to AD cutting DC in C.

Then ABCD is the parallelogram required.

Proof. By const., ABCD is a parm., and it has the side AB = K, the side AD = L, and the ∠BAD = A.

Problem 17. To construct a square on a given straight line.



Let AB be the given st. line, on which a square is to be constructed.

First Method

Construction. From A, draw AP perp. to AB.

With centre A and radius AB, draw an arc cutting AP at D.

With centres B and D and radius AB, draw two arcs intersecting at C.

Join BC, DC.

Then ABCD is the square required.

Proof. Join BD.

Then in the △ˢ CDB, ABD,

(i) CD = AB, [*Const.*

(ii) CB = AD, [*Const.*

and (iii) DB is common ;

∴ the triangles are congruent ;

so that the ∠CDB = the ∠ABD,

and the ∠CBD = the ∠ADB ;

∴ CD is parallel to AB and CB to AD.

Hence ABCD is a parm.

And in it the ∠A is a rt. angle ; ∴ it is a rectangle ;

also the adjacent sides AB, AD are equal ; [*Const.*

∴ ABCD is a square.

SECOND METHOD

Construction. From A, draw AP perp. to AB. With centre A and radius AB, draw an arc cutting AP at D.

Through D, draw DC ∥ to AB, and through B, draw BC ∥ to AD, intersecting DC at C.

[The parallels are drawn by making the ∠CDB = the ∠ABD, and the ∠CBD = the ∠ADB.]

Then ABCD is the square required.

[Draw figure and supply proof.]

Question. What are the five parts that are here given ?

EXERCISES XXXI.

[*Graphical.*]

1. Construct a quadrilateral ABCD, having given AB = 3·5 cm., BC = 6·3 cm., CD = 5·6 cm., DA = 1·7 cm., and ∠B = 70°.

2. Construct a quadrilateral ABCD, having given AB = 4·5 cm., BC = 4·8 cm., CD = 7·5 cm., ∠B = 100° and ∠C = 105°. Measure AD.

3. Construct a square on a straight line 4·7 cm. Measure both diagonals and the angles they form at their point of intersection.

4. Construct a square on a straight line $2{\cdot}3''$ as diagonal.

5. Construct a rectangle **ABCD**, given the diagonal $\mathbf{AC}=2{\cdot}5''$ and the $\angle\mathbf{ACD}=40^\circ$. Measure its sides.

6. Construct a square each of whose diagonals is 5 cm. Measure a side of this square.

7. Construct a quadrilateral whose sides are $1''$, $1''$, $1{\cdot}5''$, and $1{\cdot}2''$, and one of whose diagonals is $1{\cdot}8''$. Measure the other diagonal.

8. Construct a parallelogram whose longer side is 3·5 cm., and whose diagonals are 4 cm. and 5 cm. Measure the shorter side.

9 Construct a quadrilateral **ABCD**, having given $\mathbf{AB}=\mathbf{BC}=3{\cdot}8$ cm., $\mathbf{AC}=4{\cdot}1$ cm., the $\angle\mathbf{CAD}=26^\circ$ and the $\angle\mathbf{ACD}=35^\circ$. Measure **BD**.

10. Construct a quadrilateral **ABCD**, having given $\mathbf{AB}={\cdot}8''$, $\mathbf{BC}=1{\cdot}1''$, $\mathbf{AD}=1{\cdot}7''$, the $\angle\mathbf{C}=104^\circ$ and the $\angle\mathbf{DBC}=41^\circ$.

[*Theoretical.*]

11. Construct a parallelogram, having given two adjacent sides and an angle.

12. Construct a rectangle, having given one side and a diagonal.

13. Construct a rhombus, having given its diagonals.

14. Construct a rhombus, having given an angle and the perimeter.

15. Construct a parallelogram, having given a side, a diagonal and an angle through which the diagonal passes.

16. Construct a parallelogram, having given its diagonals and the angle between them.

17. Construct a parallelogram, having given one side and the diagonals.

18. Construct a quadrilateral, having given a side, two adjacent angles and the diagonals.

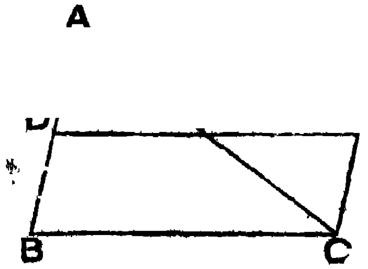
19. Construct a rhombus having given one side, and the distance between this side and the opposite side.

20. Construct a quadrilateral, equal in all respects to a given quadrilateral.

APPENDIX I

The following examples are very useful and should be as carefully studied as the Propositions that have gone before.

1. THEOREM. *The straight line, drawn through the middle point of one side of a triangle parallel to the base, bisects the other side and is half the base.* [See (XXII), 2.]



Let ABC be a triangle, and through D, the middle point of AB, let DE be drawn ∥ to BC.

To prove that DE *bisects* AC *at* E and *is half* of BC.

Through C, draw CF ∥ to AB, meeting DE produced at F.

Proof. Then BCFD is a parm.,

$\therefore DF = BC$ and $CF = BD = AD$. [I. 24.

In the $\triangle^s$ ADE, CFE,

(i) the $\angle ADE =$ the alt. $\angle CFE$, [I. 14.

(ii) the $\angle DAE =$ the alt. $\angle FCE$, [I. 14.

and (iii) $AD = CF$;

$\therefore AE = EC$ and $DE = EF$, [I. 19.

Thus AC is bisected at E.

Again $DF = BC$, [*Proved.*

$\therefore DE = \frac{1}{2}DF = \frac{1}{2}BC$. Q.E.D.

2. THEOREM. *The straight line, joining the middle points of the sides of a triangle, is parallel to the base and half of it.* [See (XXII), 7.]

Let ABC be a triangle [see Fig. of (1)]; and let D, E be the middle points of AB, AC respectively. Join DE.

To prove that DE *is parallel to* and *half* of BC.

Through C, draw CF ∥ to AB, meeting DE produced at F.

Proof. In the $\triangle^s$ ADE, CFE,

(i) the $\angle$ADE = the alt. $\angle$CFE, [I. 14.

(ii) the $\angle$DAE = the alt. $\angle$FCE,

and (iii) AE = CE ;

$\therefore$ AD = CF and DE = EF. [I. 19.

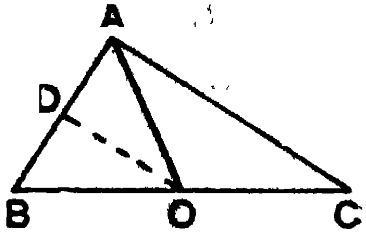
But AD = BD, $\therefore$ BD = CF.

Hence DF is equal and parallel to BC. [I. 23.

And DE = $\frac{1}{2}$DF = $\frac{1}{2}$BC. Q.E.D.

3. **Theorem.** *In a right-angled triangle, the straight line joining the middle point of the hypotenuse to the right angle is half the hypotenuse.*

Let ABC be a $\triangle$ rt. angled at A ; and let O be the middle pt. of the hypot. BC.



To prove that OA *is half* of BC.

Through O, draw OD $\parallel$ to AC meeting AB in D.

Proof. Then $\because$ OD is $\parallel$ to AC through O,

$\therefore$ D is the middle pt. of AB.

Also $\because$ the $\angle$A is a rt. angle,

$\therefore$ the angles at D are rt. angles.

Now in the $\triangle^s$ ADO, BDO,

(i) AD = BD, (ii) DO is common

and (iii) the $\angle$ADO = the $\angle$BDO ;

$\therefore$ OA = OB. [I. 4.

Hence OA = OB = OC, *i.e.*, OA is half of BC. Q.E.D.

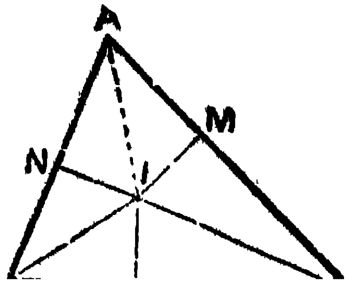
The *Converse* of this Theorem is also true, viz.,

In a triangle, if the straight line joining the middle point of a side to the opposite angle is half that side, the angle opposite to the side is a right angle.

Def. *Three or more straight lines are said to be* **concurrent** *when they all pass through one point.*

Three or more points are said to be **collinear** *when they all lie on one straight line.*

4. THEOREM. *The bisectors of the angles of a triangle are concurrent.*



Let ABC be a triangle.

Let BI and CI bisect the ∠s ABC, BCA, and meet at

Join AI.

To prove that AI *bisects the* ∠BAC.

From I, draw IL, IM, IN perp. to BC, CA and AB.

Proof. In the Δs BNI, BLI

(i) the ∠IBN = the ∠IBL,

(ii) the ∠INB = the ∠ILB, being rt. ∠s,

and (iii) IB is common to both ;

∴ IN = IL. [I. 19.

Similarly, from the Δs CMI, CLI ; IM = IL.

∴ IN = IM.

Now in the rt. ∠d. Δs ANI, AMI,

(i) the hypotenuse AI is common,

and (ii) IN = IM.

∴ the ∠IAN = the ∠IAM. [I. 20.

∴ AI is the bisector of the ∠BAC.

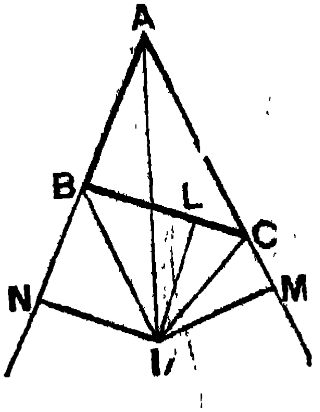
Thus the bisectors AI, BI, CI are concurrent at I.

Q. E. D.

Cor. *The point* I *is equidistant from the sides of the triangle* ABC.

For IL = IM = IN.

5. THEOREM. *The bisectors of two exterior angles of a triangle and the bisector of the third interior angle are concurrent.*



Let ABC be a △ whose sides AB, AC are produced.

Let BI_1, CI_1 be the bisectors of the exterior angles at B and C, meeting at I_1.

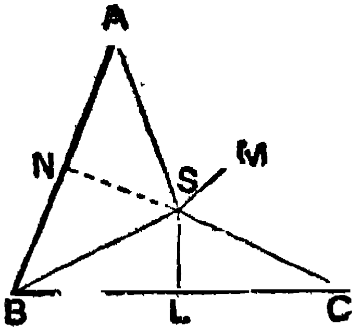
To prove that AI_1, *bisects the* ∠BAC

From I_1, draw I_1L perp. to BC and I_1M, I_1N, to AC and AB produced.

[The proof is exactly similar to that of Theorem (4).]

Cor. *The point* I_1 *is equidistant from the sides of the triangle* ABC.

6. THEOREM. *The perpendiculars to the sides of a triangle at their middle points are concurrent.*



Let ABC be a △, and L, M, N the middle pts. of BC, CA and AB.

Draw LS, MS perp. to BC, AC, meeting at S. Join NS.

To prove that NS *is perpendicular to* AB.

Proof. Join SA, SB, SC.

Now in the △^{s} BLS, CLS,

(i) BL = CL, (ii) LS is common

anp (iii) the ∠BLS = the ∠CLS, being rt. ∠^{s};

∴ BS = CS. [I. 4.

Similarly, from the △^{s}, AMS, CMS ; AS = CS

∴ AS = BS.

Again in the △ˢ ANS, BNS,

(i) AN = BN, (ii) AS = BS,

and (iii) NS is common to both ;

∴ the ∠ANS = the ∠BNS. [I. 7.

But these are adjacent angles,

∴ each is a right angle.

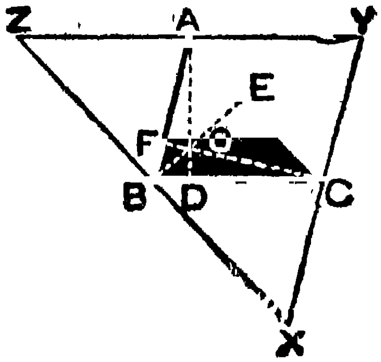
Hence NS is perp. to AB at its middle pt. N.

Q. E. D.

Cor. *The point* S *is equidistant from the angular points of the triangle* ABC.

For AS = BS = CS.

7. THEOREM. *The perpendiculars to the sides of a triangle from the opposite vertices are concurrent.*



Let ABC be a △, and AD, BE, CF the perpendiculars to BC, CA, AB from A, B and C.

To prove that AD, BE, CF *are concurrent.*

Through A, B, C, draw YZ, ZX, XY ‖ to BC, CA and AB respectively, forming the △ XYZ.

Proof. The figs. ABCY and ACBZ are parms. ;

∴ AY = BC = AZ. [I. 24.

Thus A is the middle pt. of YZ.

Similarly B and C are the middle pts. of ZX and XY.

Again AD, being perp. to BC, is also perp. to the parallel YZ.

Similarly, BE, CF are perp. to ZX and XY.

Hence AD, BE, CF being perp. to the sides of the △XYZ at their middle pts., are concurrent. [*App.* I, (6).

∴ AD, BE, CF are concurrent.

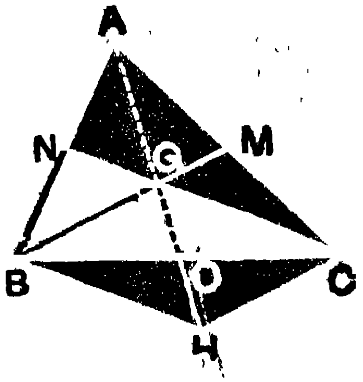
Q. E. D.

8. THEOREM. *The three medians of a triangle are concurrent.*

Let ABC be a △; and let BM, CN be two of its medians, intersecting at G.

Join AG, and produce it to meet BC in O.

To prove that AO *is the third median* of the triangle.



Construction. Through B, draw BH ∥ to CN, meeting AO produced at H

Join CH.

Proof. In the △ ABH, NG is parallel to BH through N, the middle point of AB;

∴ G is the middle point of AH. [*App.* I. (1).

Again M and G are the middle points of AC and AH;

∴ MG or BM is ∥ to CH. [*App.* I. (2).

Hence BGCH is a parallelogram;

and its diagonals bisect one another; [I. 24.

∴ O is the middle point of BC;

that is, AO is a median of the △ ABC.

Thus the medians BM, CN and AO meet at the point G.

Q. E. D.

Cor. G *is a point of* **trisection** *of each of the medians*; i. e., AG = 2GO, BG = 2GM, CG = 2GN.

For AG = GH = 2GO; &c.

Def. The point of intersection of the medians of a triangle is called the **centroid** of the triangle. Thus **G** is the centroid of the △ **ABC**. It is usual to represent the centroid by the letter **G**.

Ex. 1. Prove that **G** is the centroid of the triangle **OMN**; and if **MN** cuts **AG** at **X**, then **GX** = $\frac{1}{6}$ of the median **AO**.

Ex. 2. In any triangle. the shorter side is bisected by the greater median, and *conversely*, the greater median bisects the shorter side.

Ex. 3. A median through the intersection of any two sides makes a greater angle with the smaller side.

Ex. 3. (1) Any two sides of a triangle are together greater than twice the median to the third side.

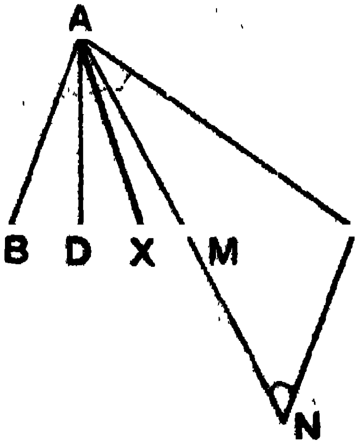
Ex. 4. Prove that in an equilateral triangle **I**, **S**, **O** and **G** are coincident.

Ex. 4. (1) The sum of the three medians of a triangle is less than the perimeter but greater than half the perimeter.

Ex. 5. Any two medians of a triangle are together greater than the third.

For **BG** + **BH** > **GH**.

Ex. 5. (1) If the sides of a triangle are unequal and from the vertex three straight lines are drawn to the base, *viz.*, the bisector of the vertical angle, the median, and the perpendicular, then the first is intermediate in position and magnitude to the other two.



Let **ABC** be △ in which **AC** > **AB**. Let **AX** be the bisector of the ∠**A**, **AM** the median and **AD** the perpendicular.

To prove that **AX** *is intermediate in position and magnitude to* **AD** and **AM**.

Produce **AM**; and make **MN** = **AM**. Join **CN**.

Proof. It is easy to see that the △s **MAB**, **MNC** are congruent;

so that **AB** = **NC**,

and the ∠**BAM** = the ∠**CNM**.

Then in the △**ACN**, **AC** > **CN**, for **AB** = **CN**;

∴ the ∠ **CNM** > the ∠ **CAN**; ∴ the ∠**BAM** > the ∠**CAM**;

∴ the ∠**BAM** > half the ∠**A**;

thus **AX** lies within the ∠**BAM**; *i. e.*, between the lines **AM** and **AB**.

Again the ∠**B** > the ∠**C** (*Hyp.*) ;
∴ compt. of the ∠**B** < compt. of the ∠**C** ;
that is, the ∠**CAD** > the ∠**BAD** ;
∴ the ∠**CAD** > half the ∠**A** ;
but **AX** lies within the ∠**CAD** ; *i. e.*, between the lines **AD** and **AC**.

Hence **AX** lies between **AD** and **AM**.

Next **AD** is perp. to **BC**, and **AX** and **AM** are obliques ; and **AX** is nearer to **AD** than **AM** ;

∴ **AD** < **AX** and **AX** < **AM** [I. 12A] ; *i. e.*, **AX** is intermediate in magnitude to **AD** and **AM**.

5. (2) To construct a triangle, having given the three medians.

Let **X**, **Y**, **Z** be the medians from the angles **A**, **B** and **C** respectively.

Construct a △**GBH** [see Fig. of (8)] such that **GH**=$\frac{2}{3}$**X**, **GB**=$\frac{2}{3}$**Y** and **BH**=$\frac{2}{3}$**Z**. Bisect **GH** at **O** ; join **BO** and produce it to **C**, making **OC**=**OB**. Join **CG**, **CH**. Produce **HG** to **A**, making **GA**=**GH**. Join **AB**, **AC**. Then **ABC** is the △ required.

Proof. Produce **BG**, **CG** to cut **AC** in **M** and **AB** in **N**.
Now because **BC** and **GH** bisect each other at **O** ;
∴ **BGCH** is a parm. ;
∴ **GM** is ∥ to **HC** and **GN** to **BH**.

Then ∵ **G** is the middle pt. of **AH**
∴ **M** is the middle pt of **AC** and **N** of **AB**.

Hence **BM**, **CN** and **AO** are medians, and **BG**=2**GM**, **CG**=2**GN** and **AG**=2**OG**

Thus **AO**=**X**, **BM**=**Y** and **CN**=**Z**.

GEOMETRICAL ANALYSIS

In Geometry, **Synthesis** is the method in which, by means of *known* results, we arrive at some *new* result ; thus for example, by the help of *theorems already established,* we prove some new theorem. The propositions in Geometry are all arranged *synthetically.*

Geometrical Analysis is the method in which we *search for the steps* which lead to the proof of a theorem or the solution of a problem. The search is thus made :—We begin by *assuming* the required result ; then from a study of the figure, deduce step by step consequences which follow from this assumption, till finally we come to a result which affords a clue to the *construction* on which the solution depends.

After the constuction has been discovered, we retrace *our steps* which led to it, and present the construction and proof in a *synthetical* form.

Analysis is very useful in attempting difficult exercises, especially problems. Thus to solve the problem :—"*Through a given point, to draw a straight line so as to cut off equal portions from the arms of a given angle.*"

Let AOB be the given angle and P the given point, from which a st. line is to be drawn to cut off equal portions from OA, OB.

Analysis. Suppose PCD to be the required line ; so that OC = OD.

Then from the figure, it is seen that COD is an isosceles triangle.

Let M be the middle point of CD. Join OM.

In the $\triangle^s$ OCM, ODM,

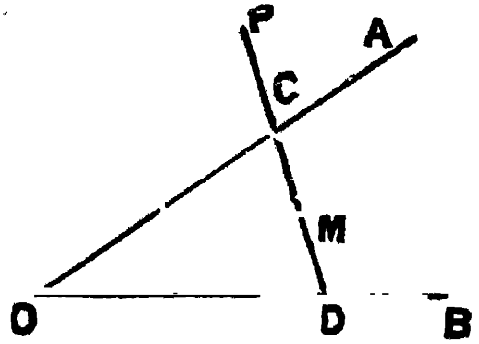
OC = OD, OM is common, and CM = DM ;

∴ the triangles are congruent ;

so that the ∠COM = the ∠DOM,

and the ∠OMC = the ∠OMD ;

thus OM **bisects the** ∠AOB, **and is perp. to** CD.



The analysis gives the following construction.

Construction. Bisect the ∠AOB by the st. line OM.

From P, draw PM perpendicular to OM, cutting OA in C and meeting OB in D.

Then PCD is the required st. line.

Proof. Now in the $\triangle^s$ OCM, ODM,

(i) the ∠COM = the ∠DOM, [*Const.*

(ii) the ∠OMC = the ∠OMD, (right angles).

and (iii) OM is common ;

∴ the triangles are congruer

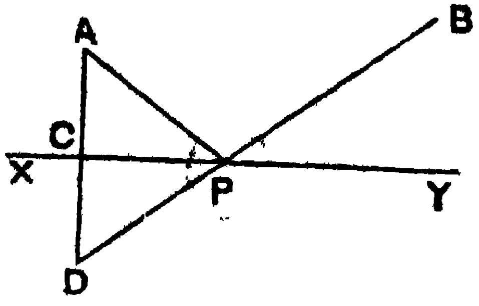
∴ OC = OD.

[The point P may be without or within the angle, the construction being the same in both cases.]

Obs. In an examination paper, the analysis should not be given, but only the construction and the proof.

In giving a construction, all its traces must be clearly shewn, though they have been omitted here.

9. PROBLEM. *To find a point in a given straight line, such that straight lines drawn to it from two given points on the same side make equal angles with the given line.*



Let **XY** be the given st. line, and A and **B** the given pts.

To find a pt. P *in* XY *such that* **AP** *and* BP *make equal angles with* **XY.**

[**Analysis.** Suppose **P** to be the required point in **XY,** such that the ∠**APX**=the ∠**BPY.**

Produce **BP** to **D,** and make **PD=PA.**

Join **AD,** cutting **XY** in **C.**

Now the ∠**APX**=the ∠**BPY**=the vert. opp. ∠**DPX.**

Thus in the △s **ACP, DCP,**

AP=DP, CP is common,

and the included ∠**APC**=the included ∠**DPC**;

∴ **AC=DC,** and the ∠**ACP**=the ∠**DCP**; and these being adj. angles are right angles.

Thus **XY bisects** AD **at right angles.**

Hence the following construction.]

Construction. From A, draw AC perp. to **XY**; produce AC to D, and make CD=AC. Join DB; and let DB cut **XY** at **P.**

Then **P** is the point required.

Proof. Join **AP.**

Then in the △s ACP, DCP,

(i) AC=DC, (ii) CP is common,

and (iii) the ∠ACP=the ∠ DCP, being rt. ∠s,

∴ the △ ACP=the △ DCP. [I. 4

∴ the ∠ APC=the ∠ DPC.

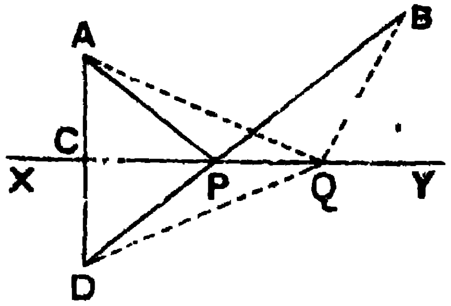
But the ∠DPC = the vert. opp. ∠BPY ;

∴ the ∠APC = the ∠BPY.

Hence AP, BP make equal angles with XY.

The construction in the case when A and B are on opposite sides of XY is obvious.

10. PROBLEM. *To find a point in a given straight line, such that the sum of its distances from two given points is a minimum.*



Let XY be the given st. line, and A, B the given pts.

To find a pt. P *in* XY *such that* AP + BP *is a minimum.*

(i) Let A, B be on the *same* side of XY.

Construction. Draw AC perp. to XY ; and produce AC to D making CD = AC.

Join DB cutting XY at P.

Then P is the point required.

Proof. Join AP. Take any other pt. Q in XY, and join QA, QB, QD.

In the △ˢ ACP, DCP,

(i) AC = DC, [*Const.*

(ii) CP is common to both

and (iii) the ∠ACP = the ∠DCP, being rt. ∠ˢ.

∴ AP = DP. [I. 4.

Similarly, from the $\triangle^s$ **ACQ, DCQ**; **AQ** $=$ **DQ**

$\therefore$ **AP** $+$ **BP** $=$ **DP** $+$ **BP** $=$ **DB**;

and **AQ** $+$ **BQ** $=$ **DQ** $+$ **BQ**.

But **DQ** $+$ **BQ** $>$ **DB**; [I. 11.

$\therefore$ **AQ** $+$ **BQ** $>$ **AP** $+$ **BP**.

And **Q** is any pt. in **XY**, other than **P**; $\therefore$ the sum of the distances of any pt. from **A** and **B** is greater than **AP** $+$ **BP**.

$\therefore$ **AP** $+$ **BP** is a minimum.

(ii) When **A** and **B** are on *opposite* sides of **XY**, then **P** is obviously the point where **AB** cuts **XY**.

Note. We see that the $\angle$**APX** $=$ the $\angle$**BPY**. Hence when **AP**, **BP** drawn from **A** and **B** make equal angles with **XY**, **their sum is a minimum.**

Ex. 6. **PAQ** is a straight line drawn through the vertex **A** of a triangle **ABC** perpendicular to the bisector of the angle **BAC**. If **Q** is any point in **PAQ**, prove that the perimeter of the triangle **ABC** is less than that of the triangle **QBC**.

Ex. 7. Of all triangles on the same base and of the same altitude, the isosceles triangle has the least perimeter.

MISCELLANEOUS EXERCISES.

1. If a straight line **AB** is bisected at **O** and produced to **P**, prove that $2OP = AP + BP$.

2. **ABC, DEF** are two triangles, such that **AB, AC** are respectively equal to **DE, DF**; if the medians that bisect **AC** and **DF** are equal, prove that the triangles are congruent.

3. The internal bisectors of two angles of a triangle can never be perpendicular to each other.

4. The bisector of the angle **A** of a triangle **ABC** meets **BC** in **D**, and **BC** is produced to **E**; prove that $\angle ABC + \angle ACE = 2\angle ADC$.

5. **ABC** is a triangle. Draw a straight line **DE** parallel to **BC**, so that **BD** and **CE** are together equal to **DE**.

6. Two triangles **ABC, ABD** stand on the same base **AB** and on the same side of it; prove that **CD** is parallel to **AB**, if $AC = BD$ and $AD = BC$.

7. In a parallelogram, the perpendiculars from a pair of opposite angles to a diagonal are equal.

8. **ABC** is an equilateral triangle and **D** is any point in **AB**; on **AD** an equilateral triangle **ADE** is drawn on the side of **AD** remote from **C**; prove that $BE = CD$.

9. **ABC** is a triangles; the angle **ACB** is bisected by **CE** which cuts **AB** in **E**; **EF** is drawn parallel to **BC** cutting **AC** in **F** and the bisector of the exterior angle at **C** in **G**; prove that $EF = FG$.

10. Through the vertex **A** of a triangle **ABC**, draw a straight line **DAE** parallel to **BC**, so that **DE** is equal to $BD + CE$.

11. If two isosceles triangles have equal vertical angles and equal altitudes, they are congruent.

12. A triangle is isosceles, if the straight lines making equal angles with the base at its extremities and meeting the opposite sides are equal.

13. On the equal sides **AB, AC** of an isosceles triangle, equilateral triangles **ADB, AEC** are drawn externally; prove that if **CD, BE** intersect at **O**, then $OD = OE$.

14. **AB, CD** bisect each other; if **AB** is greater than **CD**, prove that the $\angle ACB$ is greater than the $\angle DAC$.

15. Trisect a right angle.

16. **D**, **E** are the middle points of the sides **AB**, **AC** of a triangle; **BE**, **CD** are produced to **F**, **G**, so that **EF**=**EB** and **DG**=**DC**; shew that **FAG** is a straight line.

17. From a point equidistant from the angular points of an equilateral triangle, perpendiculars are drawn to the sides; prove that these perpendiculars are equal.

18. **P** and **Q** are the middle points of the sides of an isosceles triangle **ABC** and **R** the middle point of the base **BC**; prove that **PQ** and **AR** bisect each other at right angles.

19. The alternate sides of a pentagon are produced to meet, so as to form a star-shaped figure; shew that the sum of the internal angles at the vertices of the star is equal to two right angles.

20 Omitted.

21. Points **P** and **Q** are taken in the equal sides **AB** and **AC** of an isosceles triangle, such that **AP**=**AQ**, and **BQ**, **CP** intersect at **O**; shew that **AO**, if produced, bisects **BC** at right angles.

22. If the diagonals of a parallelogram are equal, it is a rectangle; and if they also intersect at right angles, it is a square.

23. **D** is the middle point of the base **BC** of an isosceles triangle **ABC** and **E** is any point in **AC**. Prove that the difference of **BD** and **DE** is less than the difference of **AB** and **AE**.

24. If one diagonal of a parallelogram is equal to a side, the other diagonal is greater than either side.

25 From a given point, draw a straight line making equal angles with the arms of a given angle.

26. If two medians of a triangle are equal, the triangle is isosceles; and if three medians are equal, the triangle is equilateral.

27. From any point on the bisector of an angle, parallels are drawn to its arms; prove that these parallels are equal.

28. The angle **A** is the greatest angle of a triangle **ABC**; if points **D**, **E** are taken in **AB**, **AC** respectively, prove that **BC** is greater than **DE**.

29. The bisector of the vertical angle of a triangle whose sides are unequal, divides the base into two parts of which the greater is adjacent to the greater side.

30. Place a straight line of given length between two intersecting straight lines, so as to be parallel to another given line.

31. If a triangle and a quadrilateral stand on the same base and on the same side of it, and one of them falls within the other, that which has the greater surface has the greater perimeter.

32. If two quadrilaterals have three sides of the one equal to the corresponding three sides of the other and have also the angles contained by these sides equal, the quadrilaterals are congruent.

33. If two quadrilaterals have the sides of the one equal respectively to the sides of the other and have also one angle of the one equal to the corresponding angle of the other, the quadrilaterals are congruent.

34. **ABCD** is a framework made by four rods loosely jointed together, such that **AB** = **CD** and **AD** = **BC**; if **AB** is kept fixed in position, shew that all positions of **CD** will be parallel to one another.

35. Draw a straight line parallel to the base of a triangle such that the part of it intercepted between the sides may be equal to a given straight line less than the base.

36. **AD** is a perpendicular to the base **BC** of an equilateral triangle **ABC**; prove that if an equilateral triangle be described on **AD**, its sides will be perpendiculr to those of **ABC**.

37. **ABCD** is a quadrilateral and **P**, **Q**, **R**, **S** are the middle points of its sides. If **ABCD** is a rectangle, **PQRS** is a rhombus; and if **ABCD** is a rhombus, **PQRS** is a rectangle.

38. **ABCD** is a parallelogram; **K**, **L**, **M**, **N**, are points on the sides, such that **AK** = **CL** and **BM** = **DN**; prove that **KMLN** is a parallelogram.

39. In the diagonal **AC** of a parallelogram **ABCD**, points **P**, **Q** are taken such that **AP** = **CQ**; shew that **BPDQ** is a parallelogram.

40. Three straight lines meet at a point. Draw a straight line such that the parts of it intercepted by the three lines may be equal to one another.

How many solutions has the problem?

41. In a triangle, the straight lines drawn from the extremities of the base to the opposite sides cannot bisect one another.

42. In a quadrilateral **ABCD**, the ∠**B** = the ∠**C**; prove that the ∠**A** is greater than, equal to or less than the ∠**D** according as **CD** is greater than, equal to or less than **AB**.

[*Hint.* Produce **AB** and **CD** to meet at **K**, so that **AKD** is a △.]

43. If the altitude of a triangle divide the base unequally, the greater segment is adjacent to the greater side.

44. The oblique sides **AD** and **BC** of a trapezium **ABCD** are each divided into four equal parts at **P**, **Q**, **R** and *p*, *q*, *r* respectively; prove that **P***p*, **Q***q*, **R***r* are parallel to one another.

45. Through a given point within a given angle, draw a straight line so that the part of it intercepted between the arms may be bisected at the given point.

46. **ABCD** is a parallelogram in which **AD** is double of **AB**; **AB** is produced both ways to **E**, **F**, so that **AB**=**AE**=**BF**; prove that **CE**, **DF** intersect at right angles.

47. The angles **B** and **C** of a triangle **ABC** are acute and **C** > **B**; **P** is a point in **BC**; shew that **AP** is shorter than **AB**; and find for what positions of **P**, **AP** will also be shorter than **AC**.

48. If from the vertices of a triangle three straight lines are drawn through any point within it to meet the opposite sides, their sum is greater than half the perimeter.

49. In the triangle **ABC**, the bisector of the angle **A** and the perpendicular bisector of **BC** intersect at **O**; from **O**, **OP**, **OQ** are drawn perpendicular to the sides **AB**, **AC**, produced if necessary; prove that **AP**=**AQ** and **BP**=**CQ**.

50. In a side of a triangle, find a point from which parallels drawn to the other sides are equal.

51. **ABC** is a triangle; **AD**, **BE** are perpendiculars to the opposite sides, produced if necessary; prove that the perpendicular from the middle point of **AB** to **DE** bisects **DE**.

52. **ABC** is a straight line; on **AB**, **AC** squares **ABDE**, **ACFG** are described on the same side of the straight line; shew that the perpendicular from **A** to **BG** bisects **EC**.

53. The bisectors of the angles of a triangle intersect at **I**, and **AI** produced meets **BC** in **D**; if **IL** is perpendicular to **BC**, prove that the ∠**BID**=the ∠**CIL**.

54. The sum of the distances of any point within an equilateral triangle from the three sides is **constant** (being equal to the distance of any vertex from the opposite side. [See (XXI), 12.]

55. Construct a triangle having given the base and the perpendiculars from its extremities to the opposite sides.

56. **If from either extremity of the base of a triangle, a perpendicular is drawn to the *internal* bisector of the vertical angle, the straight line joining the middle point of the base to the foot of the perpendicular is equal to half the *difference* of the two sides**

If the perpendicular is drawn to the *external* bisector, the joining line is equal to half the *sum* of the two sides.

57. Construct a triangle having given the base, an angle at the base, and the sum or difference of the sides.

58. **ABCD** is a parallelogram; **AB** is produced to **E**, so that **AB**=**BE**; if **AC** and **DE** intersect at **F**, prove that **AF**=2**CF**.

59. If the bisectors of the angles of a quadrilateral form a rectangle, shew that the quadrilateral is a parallelogram.

60. Construct a triangle, having given the middle points of the sides.

61. The straight line joining the middle points of the diagonals of a trapezium is half the difference of the parallel sides.

62. **AB**, **CD** are two parallel straight lines and **AD** is perpendicular to them; **BC** cuts **AD** in **E** and is such that **BE** is double of **AC**; prove that the ∠**ACD**=3∠**BCD**.

63. **A median of a triangle diminishes as the angle from which it is drawn increases, the sides containing the angle remaining the same.**

64. **The feet of the perpendiculars drawn from the ends of the base of a triangle to any straight line through the vertex are equally distant from the middle point of the base.**

[*Hint.* Let **BC** be the base, **O** its mid pt., **BD**, **CE** perps. to **ADE**, any line thro. **A**. Draw **OF** perp. to **DE**; produce **BD** and **OF** to cut **AC** in **G** and **H**. Prove by I. 26, that **DF**=**FE**.]

65. Omitted.

66. On the side **BC** of a triangle **ABC**, a square **BDEC** is described and through **B**, **C**, are drawn **BF**, **CF** parallel to **AD**, **AE**, respectively, meeting in **F**; prove that **AF** is perpendicular to **BC**.

BOOK II

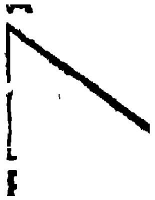
AREAS

AREAS

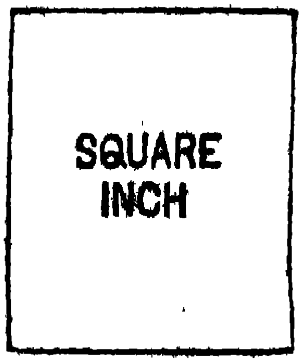
DEFINITIONS

The **area** of a plane figure is the amount of surface enclosed by its boundaries

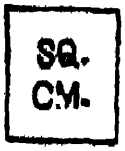
Thus the area of the △ABC is the amount of surface bounded by the sides AB, BC, AC.



A **square inch** is the area of square described on a side one inc



A **square centimetre** is the area of square described on a side one centimetre i



Similarly the terms *square foot*, *square yard*, *square mile*, *square metre*, &c., are used in the same sense.

A **unit of area** is the area of a square whose side is a unit of length.

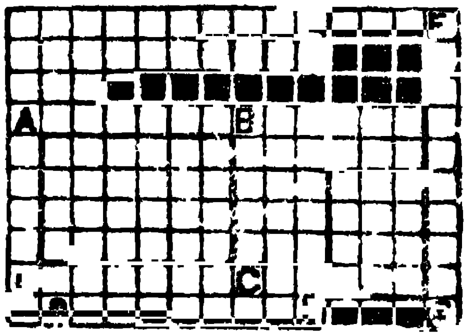
Thus if the unit of length is a *centimetre*, the unit of area is a *square centimetre*; if the unit of length is a *foot*, the unit of area is a *square foot*; &c.

To **measure** an area is to find *how often* it contains the unit of area, or *what part* it is of the unit of area.

CONGRUENT AND EQUIVALENT FIGURES

We have seen that congruent triangles are always equal in area, for they coincide with one another. Similarly congruent figures are those that coincide with one another and are therefore equal in area.

But there are figures, which though not congruent, are yet equal in area.



Thus the rectangles ABCD and EFGH drawn on squared paper are not obviously congruent, but they are yet equal in area, for each contains exactly 24 squares which are all equal to one another. Thus the word "equal" has two meanings :—(i) it means "equal in all respects or **congruent**" and (ii) it means "equal in area or **equivalent**."

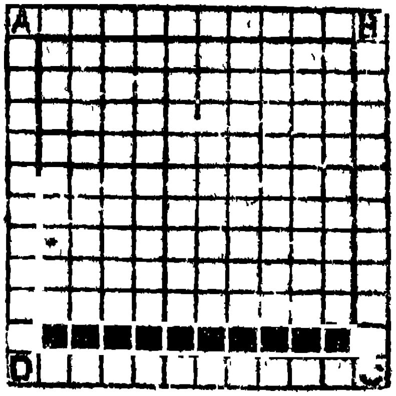
SQUARED PAPER

Squared Paper is **paper ruled into small squares** by two sets of equidistant parallel straight lines, one set being perpendicular to the other.

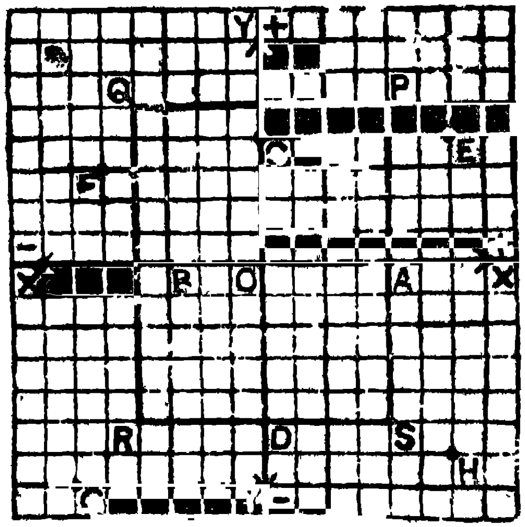
When the interval at which the parallels are drawn is a *tenth* of an inch, the paper is called a *tenth-inch squared paper* ; when the interval is a *millimetre*, the paper is a *millimetre squared paper* ; and so on.

The square ABCD, whose side is one inch, is drawn on a tenth-inch squared paper. In ABCD, there are 10 rows of squares and in each row, there are 10 squares; so that ABCD contains altogether 100 equal squares.

Hence *the area of each square on a tenth-inch squared paper is* $\frac{1}{100}$ of a sq. in. = ·01 sq. in.



AXES—CO-ORDINATES



Draw on a squared paper, two thick straight lines XOX' and YOY' perpendicular to one another.

These lines are called **axes of reference** or simply **axes**, and the point O where they intersect is called the **origin.** Moreover XOX' is called the **X-axis** and YOY' the **Y-axis**

The *four* compartments into which the axes divide the plane of the paper, are called **quadrants,** XOY being the *first* quadrant, YOX' the *second*, X'OY' the *third*, and Y'OX the *fourth.*

If O be the *starting point*, then to come to P, we may go 4 divisions of the paper along OX towards X, *i. e., to the right* and then 5 divisions *upwards*. Thus the numbers 4 and 5 determine the point P, which is called the point (4, 5). The numbers 4 and 5 are called the **co-ordinates** of the point P, 4 being termed the **abscissa** and 5 the **ordinate.**

Similarly to go to Q from, O we may travel 4 divisions along OX′ towards X′, *i. e., to the left* and then 5 divisions *upwards*. Here also 4 and 5 determine the point Q.

Again to reach R from O, we may travel 4 divisions along OX′ towards X′ and then 5 divisions *downwards*; and to reach S, we may go 4 divisions along OX towards X and then 5 *downwards*. Thus R and S are also determined by the numbers 4 and 5 as before.

Now to distinguish between these *four* cases, distances towards OX, *i. e., to the right* are considered *positive* and those towards OX′, *i. e., to the left, negative*; and distances *upwards* are *positive*, while those *downwards* are *negative*. Thus the 4 points are P (4, 5), Q (−4, 5), R (−4, −5) and S (4, −5).

Hence the **signs of co-ordinates** in the *first quadrant* are +, +; in the *second*, −, +; in the *third*, −, −; and in the *fourth*, +, −.

The four cases may be summed up thus :—

To the right is considered +, and to the left −.

Upwards is considered +, and downwards −.

To plot a point whose co-ordinates are given is to mark it on squared paper.

Area of rectangle PQRS. By counting the squares that make up the rectangle PQRS, we see that they are 80 in number. Thus the area of PQRS is 80 squares of the paper.

The area of a square on a tenth-inch squared paper is ·01 sq. in. Hence supposing the above diagram drawn on a tenth-inch squared paper, the area of PQRS = ·80 sq. in.

We obtain the number 80 thus :—(i) We see that there are 10 rows of squares and each row has 8 squares; so there are altogether 10 × 8 or 80 squares; (ii) We multiply the number of divisions (10) in the length **PS** by the number of divisions (8) in the breadth **PQ**.

Thus the expression for the area of a rectangle is stated briefly thus:—*The area of a rectangle = length × breadth.*

EXERCISES XXXII.

1. In the above figure, write down the co-ordinates of the following points, the side of a square being a unit of length : **A, B, C, D, E, F, G, H** and **O.**

2. Plot the following points : (5, 10), (3, 8), (−2, 3), (6, 6), (8, −11), (−5, 4), (0, 0,) (−6, −8), (12, −6), (8, 12), (−10, −12).

3. Plot the following points : (0, 4), (3, 4), (4, −3), (4, 0), (−3, 5), (−4, −3), (−4, 0), (5, −6), (4, 6), (0, −4), (3, −5). (−5, 6).

4. Plot the points (10, 0), (8, 0), (5, 0), (0, 0), (−2, 0), (−3, 0), (−5, 0), (−8, 0). On what straight line do these points lie ?

5. Plot the points (0, −8), (0, −5), (0, −3), (0, 0), (0, 1), (0, 4), (0, 5), (0, 6), (0, 10). On what straight line do these points lie ?

6. Plot the points (5, 4), (5, 3), (5, 2), (5, 1), (5, 0), (5, −1), (5, −2), (5, −5), and shew that they all lie on a straight line. What straight line ?

7. Plot the points (8, −4), (5, −4), (2, −4), (0, −4), (−1, −4), (−5, −4), and shew that they lie on a straight line. What straight line ?

8. Plot the points (−3, 6), (−3, 5), (−3, 1), (−3, 0), (−3, −2), (−3, −4), and shew that they lie on a straight line. What straight line ?

9. Plot the points (5, 7), (3, 7), (0, 7), (−1, 7), (−2, 7), (−4, 7), (−6, 7), and shew that they lie on a straight line. What straight line ?

Plot, on a tenth-inch squared paper, the following points, and find the area of the rectangle formed by joining them.

10. (0, 0), (18, 0), (0, −10) and (18, −10).

11. (4, 1), (4, 9), (11, 1) and (11, 9).

12. (−5, 4), (−11, 4), (−5, −7) and (−11, −7).

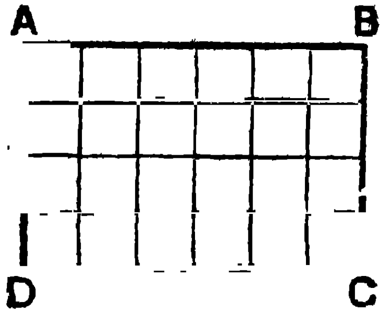
13. (3, 8), (−5, 8), (3, −5) and (−5, −5).

14. If the points in Exercises 6, 7, 8 and 9 are plotted on a tenth-inch squared paper, find the area of the rectangle formed by the straight lines.

RECTANGLES

Area of rectangle. We shall now give a formal proof for finding the area of a rectangle.

I. *When the measure of each side is an integer*, (*i.e.*, when each side contains an exact number of units).



Let ABCD be a rectangle whose side AB contains 6 units of length and AD, 4 units of length.

Divide AB into 6 equal parts and AD into 4 equal parts ; and through the points of division of each, draw parallels to the other.

Because each of the divisions of AB and AD is a unit of length,

$\therefore$ each of the figures into which the rect. ABCD is divided is a square whose side is the unit of length and therefore each square is a unit of area.

But there are 4 rows of squares and each row contains 6 squares ;

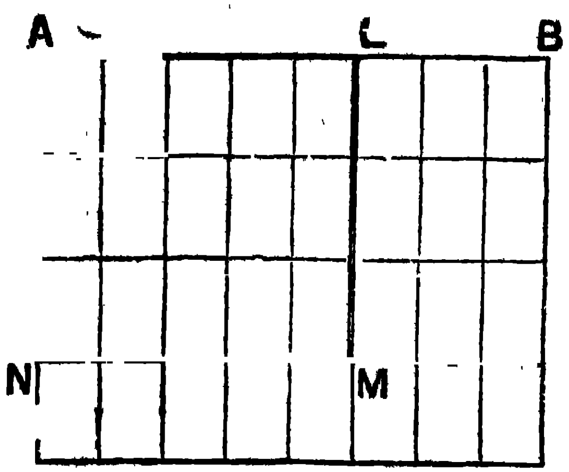
$\therefore$ ABCD contains 6×4 or 24 units of area,

that is, the area of ABCD $=$ 24 units of area.

Similarly, if AB contains a linear units and AD contains b linear units, then

area of ABCD $= (a \times b)$ units of area $= ab$ units of area.

II. *When the measure of each side is a vulgar fraction.*



Let ABCD be a rectangle, whose side AB contains $\frac{8}{5}$ units of lengths, and AD $\frac{4}{3}$ units of length.

Then the area of ABCD is $\frac{8\times 4}{5\times 3}$ or $\frac{32}{15}$ units of area.

Divide AB into 8 equal parts.

Since AB contains 8 fifth-parts of the unit,

∴ each parts is $\frac{1}{5}$ of the unit of length.

Divide AD into 4 equal parts.

Since AD contains 4 third-parts of the unit,

∴ each part is $\frac{1}{3}$ of the unit of length.

Through the points of division of each, draw parallels to the other.

Because each of the divisions of AB is $\frac{1}{5}$ unit and that of AD $\frac{1}{3}$ unit,

∴ each of the rectangles into which ABCD is divided, has one side $=\frac{1}{5}$ unit and the adjacent side $=\frac{1}{3}$ unit;

∴ all these rectangles are equal to one another.

Again AL contains 5 fifth-parts of the unit;

∴ AL is the unit of length.

For a similar reason, AN is the unit of length.

Hence ALMN is a square, having each side equal to the unit of length ;

∴ the square ALMN is the unit of area.

But ALMN contains 5×3 or 15 rectangles ;

∴ the area of each rectangle $= \frac{1}{15}$ of the unit of area.

Now ABCD contains 8×4 or 32 rectangles ;

∴ ABCD contains $32 \times \frac{1}{15}$ unit of area,

i.e., the area of ABCD $= \frac{32}{15}$ units of area.

Similarly, if AB contains $\frac{a}{b}$ units of length and AD contains $\frac{c}{d}$ units of length, then

the area of ABCD $= \frac{ac}{bd}$ units of area.

Hence the area of a rectangle, whose length is m linear units and breadth is n linear units (where m and n may be integers or fractions) is mn units of area.

This statement is often abbreviated thus :—*The area of a rectangle = length* × *breadth* (*i.e.*, the product of two adjacent sides).

The expression "*product of two sides*" means "product of the *number of units* in one side by the *number of units* in the other."

Def. If AB, AD are the adjacent sides of a rect. ABCD, then the rect. ABCD is said to be contained by AB, AD, and is denoted by "rect. AB, AD" or by "AB.AD."

Cor. 1. *The area of a square* $= a \times a = a^2$, where a denotes the sides of the square.

A square whose side is AB is denoted by "sq. on AB" or by AB^2.

Cor. 2. *The area of a right-angled triangle is half the product of the sides containing the right angle.*

For a diagonal divides a rectangle into two equal right-angled triangles, whose sides are the adjacent sides of the rectangle and the area of a right-angled triangle = ½ that of the rectangle.

Remark. The student should carefully notice that the area of a rectangle whose length is 6 in. and breadth 4 in. is (6 × 4) sq. inches and not 6 in. × 4 in. *which has no meaning*, for we cannot *multiply by 4 inches which is a concrete quantity*.

EXERCISES XXXIII.

1. Find the area of a rectangle whose adjacent sides are

   (i) 16′ 8″ and 9′ 6″ in sq. in., and also in sq. ft.

   (ii) 15 mm. and 12 mm., in sq. mm., and also in sq. cm.

   (iii) 9·3 cm. and 7·5 cm., in sq. cm., and also in sq. mm.

   (iv) 3′ 4″ and 2′ 9″, in sq. ft. and also in sq in.

   (v) 3 kilos. 25 metres and 2 kilos. 430 metres, in sq. kilos. and also in sq. cm.

   (vi) m inches and n inches, in sq. in , and also in sq. yds.

   (vii) $2a$ cm. and $3b$ cm.

2. Find the area of a square whose side is (i) 3′ 6″; (ii) 5 cm. 4 mm. ; (iii) 2 ft. (in sq. in.) ; (iv) 9 in. (in sq. ft.); (v) $3x$ cm.

3. Find the other side of a rectangle, having given

   (i) one side = 21 ft. and area = 126 sq. ft.

   (ii) one side = 5 mm. and area = 1 sq. cm.

   (iii) one side = 5½ yds. and area = 30¼ sq. yds.

   (iv) one side = a ft. and area = $4a^2$ sq. ft.

4. Find the area of a right-angled triangle, whose sides containing the right angle, are

   (i) 5·4 cm. and 3·2 cm., in sq cm. and also in sq. mm.

   (ii) 4′ 3″ and 2′ 8″, in sq. ft. and also in sq. inches.

   (iii) 105 mm. and 58 mm., in sq. mm. and also in sq. cm.

5. The area of a rectangular garden, whose sides are 200 yds. and 160 yds. is to be increased to 35000 sq. yds. By how much would the breadth be increased, if the length remains the same ?

6. Draw a diagram to shew that the area of a square whose side is $2x$ inches is four times the area of a square whose side is x inches.

7. Draw a rectangle whose adjacent sides are 2·6" and 1·8", and shew that its area is 4·68 sq. inches. (Use a tenth-inch squared paper.)

8. Draw a figure to shew that the area of a rectangle whose sides are 2 ft. 1 in. and 1 ft. 5 in. is 425 sq. inches. (Scale 1 inch to 1 foot.)

9. Find the area of a rectangular piece of land of which a plan measures 3·5 cm. by 2·6 cm. (Scale 1 mm. to 10 yds.)

10. In a plan the sides of a rectangle are shewn to be 4·8" and 3·2". If the area of the rectangle is 1536 sq. inches, on what scale is the plan drawn ?

11. One of the adjacent sides of a rectangular tennis lawn is double of the other and its area is 8450 sq. ft. Find the sides.

12. How many tiles, each 1½ ft. square, will be required to pave a court 20 ft. by 16 ft. 6 in. ?

13. *A* wishes to exchange a piece of land measuring 75 ft. by 45 ft. with *B*'s land which is in the form of a square whose side is 60 ft. ; if the price of 1 cottah of land be Rs. 800, how much has one to pay to the other ? (1 cottah = 720 sq. ft.)

14. Find the number of squares contained by a triangle whose corners are

(i) (5, 3), (10, 3), (5, 11).

(ii) (4, −5), (−4, 5), (−4, −5).

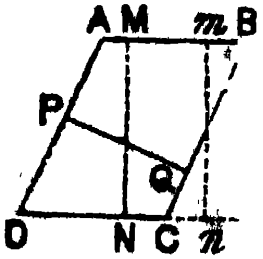
(iii) (0, 0), (0, 8), (7, 0).

DEFINITIONS

1. The **altitude** or **height** of a parallelogram is the perpendicular distance between a pair of parallel sides.

Either of the sides to which an altitude is drawn is called the **base**.

Thus **MN** or *mn* is the altitude of **ABCD**, when **AB** or **CD** is taken as base, and **PQ** is the altitude when **AD** or **BC** is taken as base.



2. The **altitude** or **height** of a triangle is the perpendicular drawn from a vertex to the opposite side, considered as base.

Thus **AD** is the altitude of the **△ABC**, when **BC** is taken as base. The altitude may fall on the base as in Fig. 1, or on the base produced as in Fig. 2.

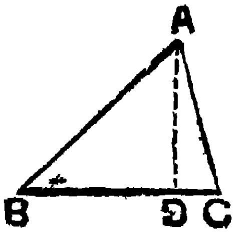


Fig. 1.

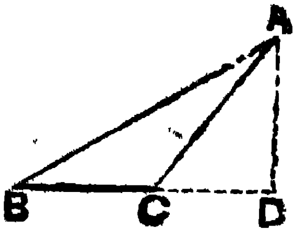


Fig. 2.

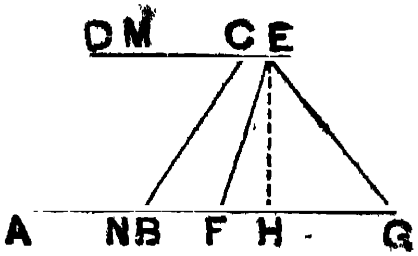
3. Parallelograms and triangles are said to be **between the same parallels**, when their bases are on one of two parallel straight lines and their opposite sides or vertices on the other.

Thus in the figure of Theorem 3, parallelograms and triangles are between the same parallels **AB** and **EF**.

LEMMA

A parallelogram and a triangle of the same altitude and on the same side of the straight line which contains their bases, are between the same parallels.

Conversely, a parallelogram and a triangle between the same parallels are of the same altitude.



Let the parm. ABCD and the △EFG be of the same altitude MN or EH; and let them be on the same side of AG which contains their bases AB and FG.

To prove that *they are between the same parallels.*

Proof. Because MN and EH are both perp. to AG;

∴ MN is parallel to EH.

And by hypothesis, MN = EH.

∴ ME is equal and parallel to NH;

that is, DE is parallel to AG.

Conversely, let the parm. ABCD and the △EFG be between the same parallels DE and AG; and let MN and EH be their altitudes.

To prove that MN = EH.

Proof. Because MN and EH are both perp. to AG;

∴ MN is parallel to EH.

But by hypothesis, DE is parallel to AG.

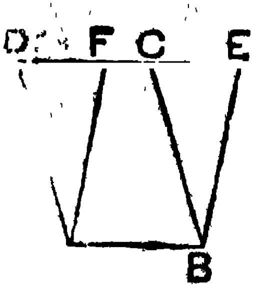
∴ MNHE is a parallelogram;

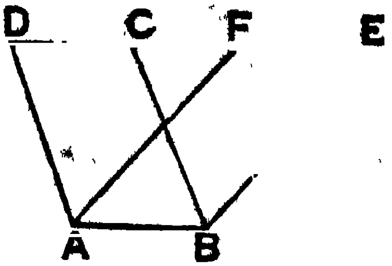
∴ MN = EH.

Q.E.D.

THEOREM 1. [Euc. I. 35.]

Parallelograms on the same base and of the same altitude are equal in area.





Let the parallelograms ABCD, ABEF be on the same base AB and of the same altitude.

To prove that *the parallelograms* ABCD *and* ABEF *are equal in area.*

Proof. Let the parms. be on the same side of AB; thus they are between the same parallels AB, DE. [*Lemma.*

Because BC is ∥ to AD and BE to AF,

∴ the ∠ BCE = the corresponding ∠ ADF,

and the ∠ BEC = the corresponding ∠ AFD,

Also BC = AD, being the opposite sides of the parm. ABCD.

Thus in the $\triangle^s$ BCE, ADF,

(i) the ∠ BCE = the ∠ ADF,
(ii) the ∠ BEC = the ∠ AFD,
and (iii) BC = AD; *Proved.*

∴ the △ BCE = the △ ADF. [I 19.

Now from the fig. ABED, take the △ BCE; the remainder is the parm. ABCD.

And from the same fig. ABED, take the △ ADF; the remainder is the parm. ABEF.

∴ these remainders are equal;

that is, the parm. ABCD = the parm. ABEF.

Q.E.D.

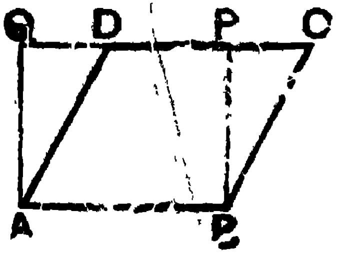
Cor. 1. *Parallelograms on equal bases and of the same altitude are equal in area.*

[Place the parms. so that they may be on the same base and on the same side of it.]

Cor 2. *A parallelogram and a rectangle on the same base and of the same altitude are equal in area.*

For a rectangle is a parallelogram.

To find the area of a parallelogram.



Let the parm. ABCD and the rect. ABPQ be on the same base AB and between the same parallels AB, QC ; so that the altitude of the parm. is AQ or BP.

Area of parm. ABCD = area of rect. ABPQ = AB. AQ.

Hence **the area of a parallelogram** is the product of the base and the altitude.

EXERCISES XXXIV.

[*Numerical and Graphical.*]

1. Draw a parallelogram equivalent to a given rectangle on the same base and having one of its angles equal to 30°.

2. Construct a parallelogram of base 3 cm. and altitude 2·5 cm., and angle 60°. Construct another parallelogram of equal area and side = 3·2 cm.

3. A parm. has its base = 1·2", the adj. side = ·6", and the angle = 50°. Construct an equivalent rhombus on the same base.

4. Construct a parm. equivalent to the parm. of Ex. 3, one of whose diagonals = 2".

5. A parm. being given, construct another of equal area on the same base and having a given angle. (Protractor is not to be used.)

6. Construct a rectangle of sides 2 cm. and 3·4 cm. Construct an equivalent parm. of sides 1·8 cm. and 4 5 cm.

[First construct an equivalent parm. on the side 3·4 cm. as base and with side 4·5 cm. Then construct an equivalent parm. on the side 4·5 cm. as base with side 1·8 cm.]

7. Draw a rhombus of area 10 sq. cm. and sides 3·5 cm.

8. Draw a parm. whose area is 15 sq. cm. and sides 3·8 cm. and 4·5 cm.

9. Find the area of the parm. whose sides are

(i) 2″ and 2·8″, and angle 60°.

(ii) 4·5 cm. and 3·4 cm., and angle 45°.

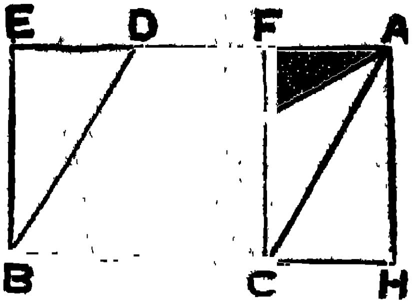
(iii) 5·6 cm. and 2·1 cm., and angle 120°.

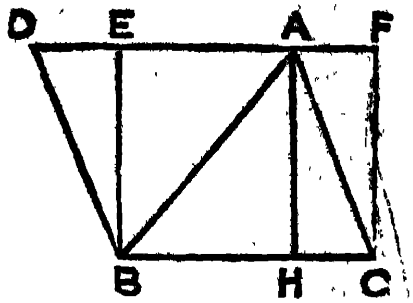
(iv) 8 cm. and 5 cm., and angle 30°.

10. Find the area of a rhombus whose side is 4 cm. and angle 60°.

* THEOREM 2.

A triangle is equal in area to half the rectangle on the same base and of the same altitude.





Let the △ ABC and the rect. BCFE be on the same base BC and of the same altitude AH ; so that they are between the same parls. BC, EA.

To prove that *the triangle* ABC *is half the rect.* BCFE.

Construction. Through B, draw BD parallel to AC to meet AE in D.

Proof. By construction, ACBD is a parm. ; and the parm. ACBD and the rect. BCFE are on the same base BC and of the same altitude AH ;

∴ the parm. ACBD = the rect. BCFE. [II. 1, *Cor.* 2.

But the △ ABC = $\frac{1}{2}$ the parm. ACBD. [I. 24.

∴ the △ ABC = $\frac{1}{2}$ the rect. BCFE.

Q. E. D.

Obs. This theorem proves that

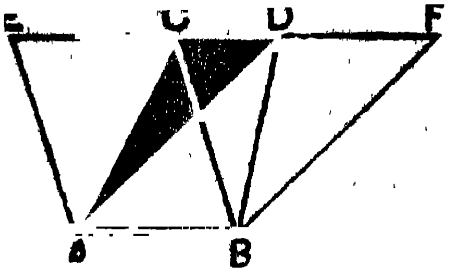
The area of a triangle is half the product of the base and the altitude.

For the area of the rect. **BCFE = BC.CF = BC.AH ;**

∴ the area of the △ **ABC = $\frac{1}{2}$ BC.AH.**

THEOREM 3. [Euc. I. 37.]

Triangles on the same base and of the same altitude are equal in area.



Let the triangles ABC, ABD be on the same base AB and of the same altitude.

To prove that *the triangles* ABC, ABD *are equal in area.*

Construction. Let the triangles be on the same sides of AB ; thus they are between the same parallels AB, CD. [*Lemma.*

Through A, let AE be drawn ∥ to BC, meeting DC produced at E.

Through B, let BF be drawn ∥ to AD, meeting CD produced at F.

Proof. By construction, ABCE, ABFD are parms ; and they are on the same base AB and of the same altitude ;

∴ the parm. ABCE = the parm. ABFD. [II. 1.

But the △ ABC = ½ the parm. ABCE, and the △ ABD = ½ the parm. ABFD ; } [I. 24.

∴ the △ ABC = the △ ABD in area.

Q. E. D.

Cor. 1. *Triangles on equal bases and of the same altitude are equal in area.*

Cor. 2. *A triangle is half of a parallelogrm on the same base and of the same altitude.*

Let the △**ABC** and the parm. **ABFD** be on the same base and of the same altitude. Join **BD**.

Then the △**ABC** = the △**ABD** = ½ the parm. **ABFD**.

Cor. 3. *A triangle is half of a parallelogram, when they are on equal bases and of the same altitude.*

EXERCISES XXXV.

1. Construct a triangle and find its area, when the
   (i) sides are 4, 5 and 6 cm.
   (ii) sides are 1·8″, 2·4″ and 3·6″.
   (iii) sides are 6, 8 and 10 cm.
   (iv) sides are ·8″, 1·5″ and 1·7″.

2. Plot on squared paper the following points and find the area of the triangle formed by joining them
   (i) (6, 6), (9, −2) and (−4, −2).
   (ii) (−2, 8), (−8, −5) and (12, −5).
   (iii) (6, 0), (3, −11) and (0, −11).
   (iv) (0, 0), (0, 7) and (−10, 0).

3. Having given a triangle,
   (i) Construct an equivalent parallelogram having an angle of 60°.
   (ii) Construct on the same base an equivalent isosceles triangle.
   (iii) Construct on the same base an equivalent right-angled triangle.
   (iv) Construct on the same base an equivalent triangle having an angle of 30°.

4. **ABCD** is a trapezium in which **AB** is parallel to **CD**; **AC**, **BD** intersect at **O**; shew that
   (i) the △**ADC** = the △**BDC**, (ii) the △**ABD** = the △**ABC**,
   and (iii) the △**AOD** = the △**BOC**.

5. **Each of the medians of a triangle bisects the triangle.**

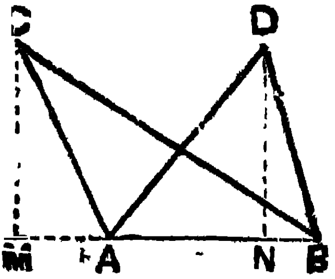
6. **AM** is a median of a triangle **ABC**; prove that if **P** is any point in **AM**, or **AM** produced, the △**PMB** = the △**PMC** and the △**PAB** = the △**PAC**.

7. Prove that the four triangles into which a parallelogram is divided by its diagonals are all equal in area.

8. **ABC** is a triangle; **D**, **E** are the middle points of **AB**, **AC**; **BE**, **CD** intersect at **O**; shew that the △**BOC** = the quadrilateral **ADOE**.

THEOREM 4. [Euc. I. 39.]

Triangles of equal area on the same base, or on equal bases are of the same altitude.



Let the △ˢ ABC, ABD be of equal area and on the same base AB.

Let CM, DN be their altitudes.

To prove that CM = DN.

Proof. The $\triangle ABC = \frac{1}{2} AB.CM$. } [II. 2.
The $\triangle ABD = \frac{1}{2} AB.DN$. }

But the $\triangle ABC =$ the $\triangle ABD$. [*Hyp.*

$\therefore \frac{1}{2} AB.CM = \frac{1}{2} AB.DN$.

$\therefore CM = DN$.

The proof is exactly the same, when the △ˢ ABC, ABD are on equal bases.

Q.E.D.

Cor. 1. *Triangles of equal area, on the same base and on the same side of it, are between the same parallels.*

Cor. 2. *Triangles of equal area, on equal bases in the same straight line and on the same side of it, are between the same parallels.*

Cor. 3. *Triangles of equal area and equal altitudes are on equal bases.*

EXERCISES XXXVI.

1. Prove by Theorems 3 and 4 that the straight line joining the middle points of the sides of a triangle is parallel to the base.

2. **ABCD** is a parallelogram ; perpendiculars **AM, CN** are drawn to the diagonal **BD** ; prove that **AM = CN**.

3. In Ex. 2, if **P** is any point in **BD** produced, prove that the $\triangle$**ABP** = the $\triangle$**CBP** and the $\triangle$**ADP** = the $\triangle$**CDP**.

4. A quadrilateral is a parallelogram, if the four triangles into which it is divided by the diagonals are equal to one another. [*Vide* I. 25.]

EXERCISES XXXVII.

1. Prove that the area of a rhombus is half the product of it diagonals.

2. If the base of a triangle is divided into any number of equal parts, the straight lines drawn from the vertex to the points of division will divide the triangle into as many equal triangles.

3. **If two triangles have two sides of the one respectively equal to two sides of the other and the included angles supplementary, the triangles are equal in area.**

4. If one diagonal of a quadrilateral bisects the other, it also bisects the quadrilateral.

5. **Two equal triangles are on the same base but on opposite sides of it ; prove that the straight line joining their vertices is bisected by the base or the base produced.**

6. The triangle formed by joining the middle point of one of the oblique sides of a trapezium to the extremities of the opposite side, is half the trapezium.

7. If a point within a parallelogram is joined to the extremities of a pair of parallel sides, the sum of the two triangles thus formed is half the parallelogram.

8. **ABCD** is a parallelogram and **P** is any point in the diagonal **AC**, or **AC** produced ; prove that the $\triangle$**APB** = the $\triangle$**APD**.

9. The join of the middle points of the sides of a triangle and any pair of parallels drawn through these points to meet the base, form a parallelogram equal to half the triangle [See (XXII), 8.]

10. The parallelogram formed by joining the middle points of the sides of a quadrilateral is half the quadrilateral. [See (XXII) 9.]

11. **Prove that, of two equivalent parallelograms, or of two equivalent triangles, (i) that which has the greater base has the smaller altitude and (ii) that which has the smaller base, has the greater altitude.**

THE THEOREM OF PYTHAGORAS

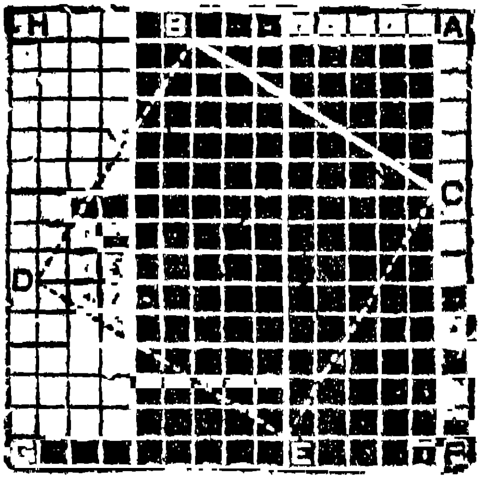
Ex. 1. Draw a △ABC right-angled at A. Let AB = 3 units and AC = 4 units. Find the area of the sq. on AB which is 3^2 or 9 units of area. Find likewise the area of the sq. on AC. Measure the hypotenuse BC and find the area of the sq. on BC. See if the last area = the first *two* areas taken together.

Ex. 2. Construct a △ whose sides containing the rt. angle are ·8" and 1·5" and repeat the process of Ex. 1.

Ex. 3. Draw a △ whose sides are AB = 5, BC = 12 and AC = 13. Measure the angle opposite to the side AC. What is its magnitude?

Find the areas of the sqs. on AB and BC and take their sum and see whether this sum is equal to the area of the sq. on AC.

Ex. 4. In the diagram, area of the sq. AG = 169 sqs. of the paper. Area of each of the △s at the corners = $\frac{1}{2} \times 5 \times 8$ or 20 sqs. of the paper. Hence area of the sq. BCED = 169 − 4 × 20 or 89 sqs. of the paper. Area of the sq. on AB = 64 sqs. of the paper; area of sq. on AC = 25 sqs. of the paper. Thus the sum of areas of the sqs. on AB, AC = 89 sqs. of the paper = area of the sq. BCED.



Mark that the angle opposite to the side BC is a rt. angle.

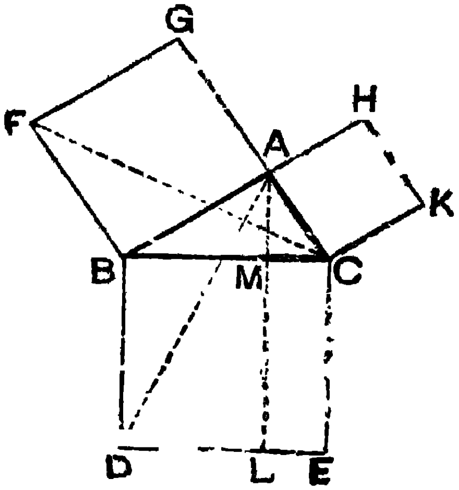
Ex 5. Draw on squared paper a figure with different numbers and repeat the process of Ex. 4.

From these examples we see that "*In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.*"

This famous theorem is usually attributed to PYTHAGORAS, a Greek philosopher, who lived between 580 and 510 B.C., and is known as the Theorem of Pythagoras. There is, however, reason to believe that it was known to the ancient Indians long before the time of the Greek philosopher.

THEOREM 5. [Euc. I. 47.]

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle. [Pythagoras' Theorem.]



Let ABC be a right-angled triangle, having the ∠ BAC a rt. angle.

To prove that

the sq. on BC = *the sq. on* AB + *the sq. on* AC.

Construction. On BC draw the square BCED, and on AB, AC draw the squares ABFG, ACKH.

Draw AL ‖ to BD or CE, to meet DE at L.

Join AD, FC.

Proof. Since the adj. ∠ˢ BAC, BAG are rt. angles,

∴ CA, AG are in the same straight line. [I. 2.

Now the rt. ∠CBD = the rt. ∠ABF;

to each add the ∠ABC;

∴ the whole ∠ABD = the whole ∠FBC.

Then in the △ˢ ABD, FBC,

(i) AB = FB [sides of a sq.]

(ii) BD = BC [sides of a sq.]

and (iii) the included ∠ABD = the included ∠FBC;

∴ the △ABD = the △FBC. [I. 4.

Now the rect. BL and the $\triangle$ ABD are on the same base BD and have the same altitude ;

$\therefore$ the rect. BL = 2 $\triangle$ ABD. [II. 2.

Also the sq. BG and the $\triangle$ FBC are on the same base FB and have the same altitude ;

$\therefore$ the sq. BG = 2 $\triangle$ FBC. [II. 2.

But the $\triangle$ ABD = the $\triangle$ FBC ; [*Proved.*

$\therefore$ the rect. BL = the sq. BG.

In the same manner by joining AE, KB, it can be shewn that the rect. CL = the sq. CH.

$\therefore$ the rect. BL + the rect. CL = the sq. BG + the sq. CH ;

$\therefore$ the sq. BE = the sq. BG + the sq. CH ;

that is, the sq. on BC = the sq. on AB + the sq. on AC.

Q. E. D

Cor. 1. From the proof it is seen that

the sq. BG = the rect. BL = the rect. BC, BM ;

and the sq. AK = the rect. CL = the rect. BC, CM ;

where M is the foot of the perp. AM from the rt. angle A.

Hence *if from the right angle of a right-angled triangle, a perpendicular be drawn to the hypotenuse, the square on a side is equal to the rectangle contained by the hypotenuse and that segment of the hypotenuse which is adjacent to that side.*

Cor. 2. From this Theorem, we have

$$BC^2 = AC^2 + AB^2.$$

Thus if the sides opposite to the angles A, B, C be respectively denoted by a, b, c units of lengh, then

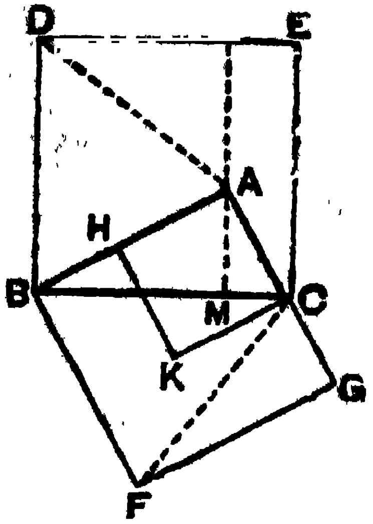
$$a^2 = b^2 + c^2$$

Hence $b^2 = a^2 - c^2$ and $c^2 = a^2 - b^2$.

Thus *any two sides of a right-angled triangle being given, the third side can be found.*

Obs. This Theorem can be proved by taking any one of the *eight* figures formed by turning the squares in all possible directions. As an example, we take the figure in which all the squares are turned inwards.

Let the squares BCED, ABFG, ACKH be drawn as in the figure.



Draw LAM ∥ to BD or CE, to meet DE at L and BC at M.

Proof. Join AD, CF.

Now the rt. ∠CBD = the rt. ∠ABF ;

from each, subtract the ∠ABC ;

∴ the remaining ∠ABD = the remaining ∠FBC.

Then in the $\triangle^s$ ABD, FBC,

(i) AB = FB [sides of a sq.],

(ii) BD = BC [sides of a sq.],

and (iii) the included ∠ABD = the included ∠FBC ;

[*Proved.*

∴ the △ABD = the △FBC. [I. 4.

Now the rect. BL and the △ABD are on the same base BD and have the same alt. ;

∴ the rect. BL = 2△ABD. [II. 2.

Also the sq. BG and the △FBC are on the same base FB and have the same alt. ;

∴ the sq. BG = 2△FBC.

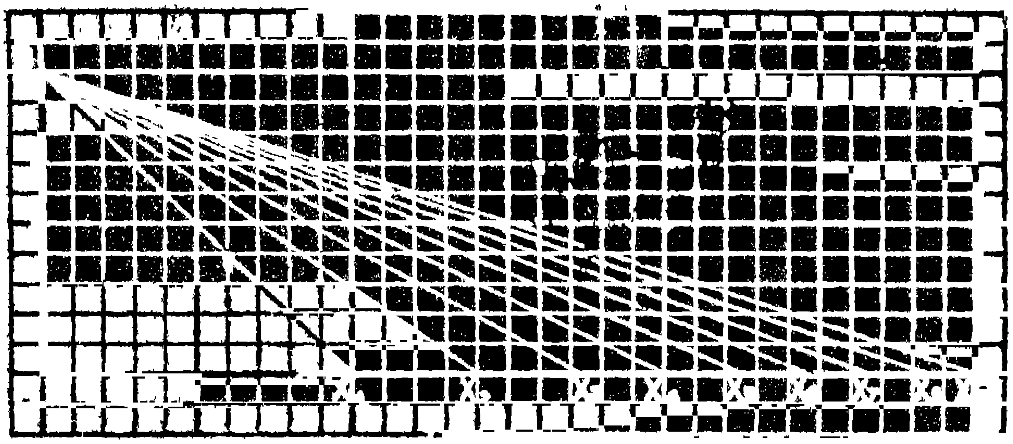
But the △ABD = the △FBC ; [*Proved.*

$\therefore$ the rect. BL = the sq. BG.

Similarly, by joining AE, BK, it can be proved that the rect. CL = the sq. CH.

$\therefore$ the rect. BL + the rect. CL = the sq. BG + the sq. CH; that is, the sq. BE = the sq. BG + the sq. CH. Q.E.D.

Note 1. To find graphically $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$,



Let AB be a st. line 1 inch long. Draw a st. line of indefinite length perp. to AB at A.

Make $AX_1 = AB$; and join BX_1.

Then $$BX_1^2 = AX_1^2 + AB^2 = 1^2 + 1^2 = 2;$$

$$\therefore BX_1 = \sqrt{2}.$$

Make $AX_2 = BX_1$; thus $AX_2 = \sqrt{2}$.

Join BX_2.

Then $$BX_2^2 = AX_2^2 + AB^2 = (\sqrt{2})^2 + 1^2 = 3;$$

$$\therefore BX_2 = \sqrt{3}.$$

Make $AX_3 = BX_2$; thus $AX_3 = \sqrt{3}$.

Similarly, if $AX_4 = BX_3$, $AX_5 = BX_4$, $AX_6 = BX_5$ then $AX_4 = \sqrt{4}$, $AX_5 = \sqrt{5}$, $AX_6 = \sqrt{6}$, ...

Hence $AX_2=\sqrt{2}$, $AX_3=\sqrt{3}$, $AX_4=\sqrt{4}$, $AX_5=\sqrt{5}, \ldots$, and by measurement,

$$\sqrt{2}=1{\cdot}41,\ \sqrt{3}=1{\cdot}73,\ \sqrt{4}=2,\ \sqrt{5}=2{\cdot}24,\ \sqrt{6}=2{\cdot}45,\ldots$$

correct to 2 decimal places.

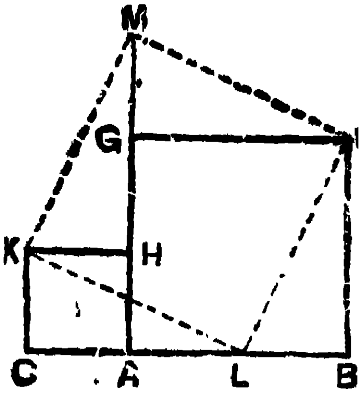
Note. 2. **To construct a square equal to twice, thrice, four times......a given square.**

The construction is evident from the above figure ; for $AX_2^2=2AB^2$, $AX_3^2=3AB^2$, $AX_5^2=5AB^2$, and so on.

Remarks

Several demonstrations of this celebrated Theorem have been given from time to time by mathematicians, some of which are such as to afford ocular proof of its truth. The following two are interesting examples, in which two squares will be *dissected* and then their parts will be put together to form a third square.

First Proof by Dissection



Draw on a piece of paper the sqs. ABFG and ACKH side by side, so that AB, AC may be in one st. line.

Make LB = AC, and join KL, LF.

Produce AG to M, making GM = AC, and join KM, MF.

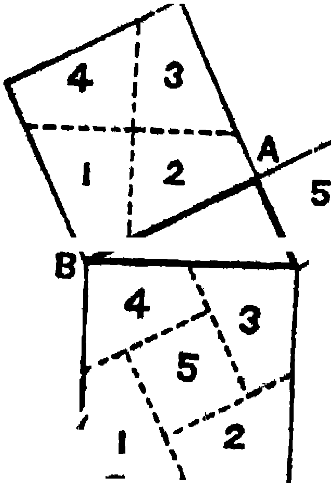
It is easily seen that the Δ^s KCL, LBF, MGF and KHM are all congruent and that the figure KLFM is a sqaure.

Now cut off from the two squares on AB, AC, *i. e.*, from the figure BFGHKC, the two Δ^s KCL, LBF, and there remains the pentagon GHKLF.

Fit to the pentagon **GHKLF**, the △ˢ **KCL**, **LBF** (which are equal to the △ˢ **MGF**, **KHM**), and the sq. **KLFM** is formed.

Thus the sq. **KLFM** = the two sqs. **ABFG**, **ACKH**, *i.e.*, the sq. on the hypotenuse **KL** = the sum of the sqs. on the two sides **CL**, **CK**.

Second Proof by Dissection



Through the middle points of the sides of the sq. on **BC**, draw parallels to **AB**, **AC**, dividing the sq. into 4 quadrilaterals marked 1, 2, 3, 4 and the sq. marked 5.

Take one of the smaller squares (that on **AB**), and through the pt. of intersection of the diagonals (not shewn in the diagram), draw two st. lines parallel and perpendicular to the hypot. **BC**. By dissecting the square along these two lines, we get the four quadrilaterals marked 1, 2, 3, 4, which will be seen to be congruent.

Cut off the sq. on **AC** marked 5.

Now these dissected parts *viz.*, 1, 2, 3, 4 and 5, of the smaller squares will be seen to fit exactly into the sq. on **BC**, whose corresponding parts are also marked 1, 2, 3, 4 and 5.

Thus the sq. on the hypotenuse **BC** = the sum of the sqs. on the sides **AB**, **AC**.

This proof is due to Henry Perigal, F. R. A. S., and is known as Perigal's "Dissection." See *The Messenger of Mathematics* (new series), Vol. II.

EXERCISES XXXVIII.

[*Numerical and Graphical*].

1. Find the hypotenuse of a right-angled triangle, given that the sides containing the right angle are

(i) 9 and 12 ; (ii) 4 and 7·5 ; (iii) 8 and 7 ;
(iv) ·5 and 1·2 ; (v) 12·3 and 14·6 ; (vi) 15·2 and 18·4

2. Find the area and the other side of a right-angled triangle, having given

(i) one side $=15$ and hypotenuse $=17$;
(ii) one side $=14$ and hypotenuse $=50$;
(iii) one side $=12{\cdot}3$ and hypotenuse $=16{\cdot}4$;
(iv) one side $=2{\cdot}4$ and hypotenuse $=14{\cdot}5$;

3. Plot on a tenth-inch squared paper, the pair of points and find the distance between them

(i) (4, 5) and (1, 1) ; (ii) (8, 12) and (−6, 11) ;
(iii) (−3, −5) and (−18, 3) ; (iv) (6, 0) and (0, 8) ;
(v) (0, 0) and (−12, 16) ; (vi) (8, 10) and (−5, 0) ;
(vii) (−18, −4) and (5, −7) ; (viii) (−10, 9) and (−8, −8).

4. Find the sides of the triangle whose vertices are (3, 4), (0, −5) and (−3, 2). [Use tenth-inch squared paper.]

5. In Fig. of Theorem 5, if **AC** $=7$ cm. and **BC** $=25$ cm., find **BM**. Also if **AB** $=12$ cm. and **BM** $=9{\cdot}6$ cm., find **AC**.

6. Find (i) the diagonal of a rectangle, when its sides are 24 and 76 ; and (ii) one side, when the other side is 35 and diagonal 37.

7. Find the diagonal of a square whose side is (i) 35, and (ii) 54·2.

8. Find the side and area of a square, whose diagonal is (i) 15 and (ii) 45.

9. Find the height and area of an equilateral triangle whose side is 12 cm.

10. Find the side of a rhombus whose diagonals are (i) 15 and 36, and (ii) 18 and 24.

11. An iron rod rests against a wall with its foot 15 ft. from the wall ; if the top of the rod is 30 ft. above the ground, how long is the rod ?

12. A man walks 60 yds. due west and then 144 yds. due north ; how far is he from the starting point ?

13. A man walks 62 yds. due south, then 35 yds. due west and lastly 50 yds. due north ; how far is he now from the starting point ?

14. A ladder rests against the top of a wall 24 ft. high with a part 4 ft. in length projecting beyond the wall. If its foot be at a distance of 10 ft. from the wall, how long is the ladder ?

15. A rope 435 ft. long is attached to the top of an upright pole. When stretched tight, it touches the ground at a point 429 ft. from the foot of the pole. Find the height of the pole.

16. A ladder 50 ft. long reaches a window 40 ft. high ; it is then turned over to the other side of the street and reaches a point 48 ft. above the ground. What is the breadth of the street ?

17. Two posts, 35 and 50 ft. high, are 20 ft. apart. What is the distance between their tops ?

[*Theoretical.*]

18. **Use this Theorem to prove I. 20.**

19. Construct a square equal to the sum of two given squares.

20. Construct a square equal to the sum of three or more given squares.

21. Construct a square equal to the difference of two given squares.

22. If the diagonals of a quadrilateral intersect at right angles, then the sum of squares on two opposite sides is equal to the sum of the squares on the other two sides.

23. **ABC** is a triangle right-angle at **A** ; **P**, **Q**, are points in **AB**, **AC** ; shew that

$$PC^2+QB^2=BC^2+PQ^2.$$

24. The sum of the squares on the sides of a rectangle is equal to the sum of the squares on its diagonals.

25. **AD** is the perpendicular from **A** to the base **BC** of a triangle **ABC**. Prove that

$$AB^2 \sim AC^2 = BD^2 \sim CD^2.$$

26. **The square on the diagonal of a square is equal to twice the square.**

27. The sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

28. **Three times the square on a side of an equilatral triangle is equal to four times the square on the perpendicular from any angular point to the opposite side.**

29. The diagonal BD of a parallelogram ABCD is at right angles to the sides AB and CD ; prove that the difference between the squares on the diagonals is four times the square on AB.

30. From any point O within an acute-angled triangle ABC, perpendiculars OP, OQ, OR are drawn to the sides BC, CA and AB respetively ; prove that

$$AR^2+BP^2+CQ^2=AQ^2+CP^2+BR^2.$$

31. Divide a given straight line into two parts, so that the square on one part may be twice the square on the other part.

32. In the Figure of Theorem 5—

(*a*) Prove that the $\triangle ABD =$ the $\triangle FBC$, by rotating the first triangle about B through a right angle. Hence shew that AD and FC intersect at right angles.

Prove similarly that the $\triangle ACE =$ the $\triangle KCB$ and that AE and KB intersect at rght angles.

(*b*) Prove that the $\triangle FBD =$ the $\triangle KCE =$ the $\triangle ABC$ [(xxxvii), 3.]

(*c*) The straight lines FG, KH and LA meet at one point T.

(*d*) If DB, EC be produced to meet FG, KH at P and Q, prove that (i) ABPT is a parallelogram and it is equal to the rect. BL, and (ii) ACQT is a parallelogram and it is equal to the rect. CL.

(*e*) Prove also that (i) ADBT is a parallelogram and it is equal to the rect. BL, and (ii) AECT is a parallelogram and it is equal to the rect. CL.

(*f*) Prove that CF is perpendicular to BT and BK to CT.

(*g*) Prove that CF, BK and AL are concurrent.

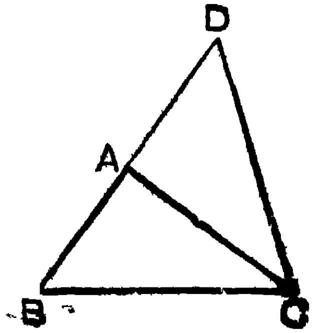
33. From the Exercises (*c*) and (*d*) above, deduce a proof of Theorem 5.

34. The angle contained by two sides of a triangle is obtuse or acute, according as the sum of the squares on these two sides is less or greater than the square on the third side.

THEOREM 6. [Euc. I. 48.]

[Converse of Theorem 5.]

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these two sides is a right angle.



Let ABC be a triangle; and let

the sq. on BC = the sq. on AB + the sq. on AC.

To prove that *the angle* BAC *is a right angle.*

Construction. Let AD be drawn perpendicular to AC.

Make AD equal to AB, and join DC.

Proof. Since the ∠DAC is a right angle;

∴ the sq on DC = the sq. on AD + the sq. on AC, [II. 5.

But AD = AB. [*Const.*

∴ the sq. on DC = the sq. on AB + the sq. on AC.

And by hypothesis,

the sq. on BC = the sq. on AB + the sq. on AC.

∴ the sq. on DC = the sq. on BC;

∴ DC = BC.

Now in the △ˢ DAC, BAC,

(i) AD = AB [*Const.*

(ii) AC is common,

and (iii) DC = BC; [*Proved.*

∴ the ∠ DAC = the ∠ BAC. [I. 7.

But the ∠ DAC is a right angle; [*Const.*

∴ the ∠ BAC is a right angle.

Q.E.D.

Note

To find **three integers** *to represent the three sides of a right-angled triangle.*

We have the *identity*

$$(m^2+1)^2 = (m^2-1)^2+(2m)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

From (i), we see that *the square of one number = the sum of the squares of two others.*

Thus

$$\left.\begin{array}{r} m^2+1 \\ m^2-1 \\ 2m \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

represent the sides of a right-angled triangle.

Also since (i) is an identity, it is satisfied by *any value whatever* of m. Hence by giving to m in succession the values 2, 3, 4, 5, 6,……, we can find an unlimited number of groups of *three integers* to represent the sides of a right-angled triangle.

Thus when $m=2$, $m^2+1=5$, $m^2-1=3$, $2m=4$;

when $m=4$, $m^2+1=17$, $m^2-1=15$, $2m=8$;

and so on. Hence the

Rule. *Take any number*, (i) *add unity to its square to obtain the hypotenuse*, (ii) *subtract unity from its square to obtain one side, and* (iii) *take twice the number to get the third side.*

[Observe that the same multiple or sub-multiple of any of the groups of numbers found from (ii) likewise serves our purpose.]

EXERCISES XXXIX.

1. **If the difference of the squares on two sides of a triangle is equal to the square on the third side, the triangle is right-angled.**

2. The sides of a triangle are as $1 : 2 : \sqrt{3}$; prove that the triangle is right-angled.

LOCI

Preliminary Exercises

Ex. 1. Mark a point at a distance of 3 cm. from a fixed point **O**. Mark 12 such points **P, Q, R, S,** .....and join them freehand. What figure is formed? See (i) that *every* point which is at a distance of 3 cm. from **O** lies on this figure; and (ii) that *any* point on this figure is at a distance of 3 cm. from **O**.

Observe that the figure formed is a circle whose centre is **O** and radius 3 cm. This circle is called the *locus* of the points **P, Q, R, S,** .....

The points P, Q, R, S ......may be considered as the *different positions of a moving point.* To mark these positions on paper is to **plot the locus** of the point.

To plot a locus, it is convenient to use **squared paper**.

Ex. 2. Mark a point **P** at a distance of 1 inch from a fixed straight line **AB**, on one side of it. Mark 8 other points **Q, R, S,** .... at the same distance from **AB**, and join them. What figure is formed? Note (i) that *every* point which is at a distance of 1 inch from **AB** is on this figure; and (ii) that *any* point on this figure is at a distance of 1 inch from **AB**.

Find 10 points *p, q, r, s,*......on the other side of **AB**, each at a distance of 1 inch from it. Shew that, if they are joined, a similar figure is formed.

Observe that the two figures are two straight lines parallel to **AB**. The pair of straight lines, one on each side of **AB**, is called the *locus* of the points **P, Q, R,** . ...., *p, q, r* ..... ; or rather the locus of a moving point whose different positions are **P, Q** ....

Ex. 3. Cut two circles out of a thin cardboard. Fix one of them on a piece of paper and slide the other along the edge of the first. Prick several points on the paper through the centre of the moving circle in its successive positions. Observe that the points pricked on the paper form a circle.

The circle is the *locus* of the centre of the moving circle. What are its centre and radius?

Ex. 4. A point moves so as to be equidistant from two parallel straight lines. Find its locus.

Ex. 5. A circular ring moves on the floor in a straight line. What is the locus of its centre?

Ex. 6. A wheel is rolled along a straight line in a plane. What is the locus of its highest point?

Ex. 7. A horse runs round a circular race-course so as to be always at the same distance from the outer edge of the course. Find its locus.

Ex. 8. A point moves so that its distance from the circumference of a given circle is constant. Find its locus.

Note. The distance of a point from the circumference of a circle is measured **radially**, *i.e.*, by the part (*intercepted between the point and the circumference*) of the radius that passes through the point.

The above exercises will lead to the following definition of a locus.

DEF. *The path traced out by a point, which moves so as to satisfy some given condition, is called the* **locus** *of the point.*

The trace of a moving point is either *straight line* or *a curve*; hence the locus of a point is a line, or part of a line, straight or curved.

In *Prem. Ex.* 1, the condition imposed is that the moving point *is to be always at a distance of* 3 cm. *from* O; and we see (i) that every point which satisfies this condition is on the circle, and (ii) that any point on the circle satisfies this condition. Hence the locus of a point which moves so as *to be always at the same distance from a given point*, is a circle.

When a 'circle' is spoken of as a locus the 'circumference' is meant.

Again, in *Prem. Ex.* 2, the given condition is that the moving point *is to be always at a distance of* 1 in. *from* AB; and we see (i) that every point which satisfies this condition is on the pair of st. lines, and (ii) that any point on them satisfies that condition. Hence the locus of a point which moves so as *to be always at the same distance from a given straight line* is a pair of straight lines parallel to the given straight line, one on each side of it.

And so on.

Thus to establish a locus *completely*, it is necessary to prove that

(i) *Every point which satisfies the given condition lies on the locus*; and

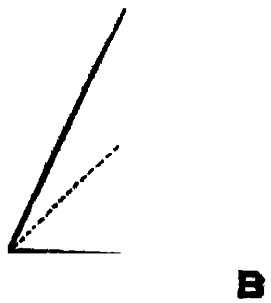
(ii) *Any point on the locus satisfies the given condition.*

These **two** things have been shewn in the case of the two important loci, discussed in Theorems 7 and 8.

But in proving a locus, it is usual to give only the first part of the proof, the second part being generally omitted.

THEOREM 7.

The locus of a point, which is equidistant from two fixed points, is the perpendicular bisector of the straight line joining the two fixed points.



Let A, B be two fixed points; and let P be a moving point such that PA = PB in all positions of P.

To prove that *the locus of* P *is the perpendicular bisector of* AB.

(i)

Let P be *any* position of the moving point;

[thus PA=PB.

Let O be the middle pt. of AB; thus O is another position of the moving point, for OA=OB.

Join OP.

Proof. Then in the $\triangle^s$ AOP, BOP,

(i) AO = BO, [*Const.*

(ii) OP is common,

and (iii) PA = PB; [*Hyp.*

∴ the ∠AOP = the ∠BOP, [I. 7.

∴ PO is perpendicular to AB.

Thus *every* point P which is equidistant from A and B, lies on OP the perpendicular bisector of AB.

(ii)

Let Q be *any* point on the perpendicular bisector of AB.

Join QA, QB.

Proof. Then in the $\triangle^s$ AOQ, BOQ,

(i) AO = BO

(ii) OQ is common,

and (iii) the $\angle$ AOQ = the $\angle$ BOQ, being rt. angles;

$\therefore$ AQ = BQ. [I. 4.

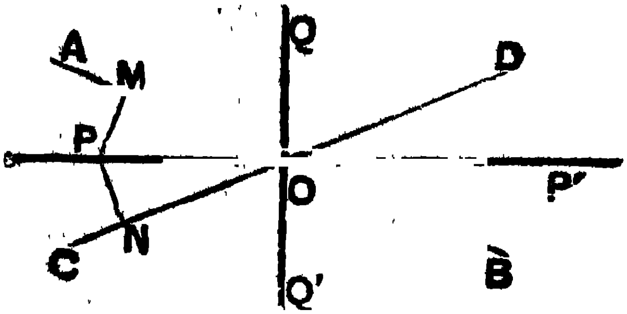
Thus *any* point Q on the perpendicular bisector of AB is equidistant from A and B.

Hence the perpendicular bisector of AB (produced infinitely both ways) is the locus required.

Q. E. D.

THEOREM 8.

The locus of a point, which is equidistant from two given intersecting straight lines, is the pair of bisectors of the angles between the given lines.



Let AB, CD be two st. lines intersecting at O; and let P be a point which moves so that in all positions of P, the perp. PM to AB = the perp. PN to CD.

To prove that *the locus of* P *is the pair of bisectors of the angles* AOC and AOD.

(i)

Let P be *any* position of the moving point within the ∠AOC; thus PM = PN.

Join OP.

Proof. Then in the right-angled Δ^s OPM, OPN,

(i) OP is the common hypotenuse,

and (ii) one side PM = one side PN; [*Hyp.*

∴ the triangles are congruent, [I. 20.

so that the ∠POM = the ∠PON.

Thus *every* point P which is equidistant from AB and CD lies on PP′, the bisector of the ∠AOC.

Similarly, if P is within the ∠AOD, it can be proved that P lies on QQ′, the bisector of the ∠AOD.

(ii)

Let P be *any* point on the bisector PP′.

Draw PM, PN perpendicular to AB and CD.

Proof. In the △ˢ POM, PON,

(i) the ∠POM = the ∠PON, [*Hyp.*

(ii) the ∠OMP = the ∠ONP, being rt. angles,

and (iii) OP is common ;

∴ the triangles are congruent ; [I. 19.

∴ PM = PN.

Thus *any* point P on the bisector PP′ is equidistant from AB and CD.

Similarly it can be shewn that any point on the bisector QQ′ is equidistant from AB and CD.

Hence the locus required is the pair of st. lines PP′ and QQ′ (of unlimited length) which bisect the angles between AB and CD.

Q.E.D.

EXERCISES XL.

1. Two diameters of a circle are at right angles to each other. Prove that any point in one of them is equidistant from the extremities of the other.

2. Prove that the middle point of a side of an equilateral triangle is equidistant from the other two sides.

3. Prove that the point of intersection of the diagonals of a rhombus is equidistant from its sides.

4. From A the vertex of an isosceles triangle, AD is drawn perpendicular to the base BC. Prove that any point in AD is equidistant from B and C and also from AB and AC.

5. Find the locus of the vertices of all triangles which have the same base and the same altitude.

6. A number of triangles stand on the same base and have their vertices on a parallel to the base. Find the locus of the middle point of their sides.

7. Straight lines are drawn parallel to the base of an isosceles triangle and terminated by the sides. Find the locus of their middle points.

8. Straight lines are drawn from a fixed point to a given straight line. Find the locus of their middle points.

9. Straight lines are drawn from a fixed point to a given circle. Find the locus of their middle points.

The point may be *outside*, *on* or *inside* the circle.

10. **AB** is a straight line of which **C** is the middle point. Find the locus of a point at which **AC**, **BC** subtend equal angles.

11. **ABCD** is a straight line of which **AB** and **CD** are equal segments. Find the locus of a point at which **AB**, **CD** subtend equal angles.

12. **A** is a given point and **P** a point that moves in a given straight line; **Q** is a point in **AP** produced, such that **AP** is always equal to **PQ**. Find the locus of **Q**.

13. Find the locus of the vertices of triangles which have the same base and medians of given length that bisect the base.

14. Find the locus of the vertices of right-angled triangles which are described on a given line as hypotenuse.

15. A ladder stands upright against a wall. If it gradually slips down till it lies on the floor, what is the locus of its middle point?

16. Find the locus of a point which moves so that the sum, or difference, of its distances from two given intersecting straight lines of unlimited length is constant.

[*Hint.* Let **AB**, **AC** be the given st. lines, and **X** the sum or difference of the distances of the moving pt., **X** being constant. Draw **AD** perp. to **AB** and make **AD** = **X**. Through **D**, draw **DE** parl. to **AB** cutting **AC** in **O**. Bisect the angles **AOE** and **AOD** by lines meeting **AB** in **F** and **G**. Then this pair of bisectors is the locus required.]

INTERSECTION OF LOCI

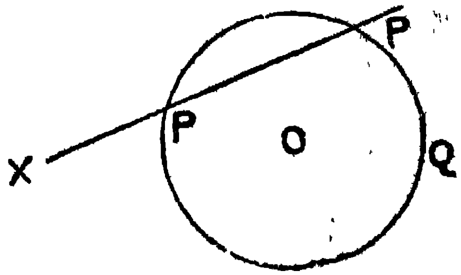
The method of Loci is useful in finding the position of a point which satisfies *two* given conditions and thus enables us to solve many geometrical problems.

Ex. 1. In a given straight line (**X**), find a point which is at a given distance from a fixed point (**O**).

Here we have to find a point which satisfies *two* conditions :—

(i) it must lie on **X** ; and

(ii) it must be at a given distance from **O**.



Evidently (i) gives **X** as the locus of the point, and

(ii) gives as its locus, the circle **PP'Q**, which is described with centre **O** and radius equal to the given distance ; for all points on this circle are at the given distance from **O**.

Now to satisfy the two given conditions, a point must lie on the line **X** as well as on the circle **PP'Q** and must therefore be the point common to both **X** and **PP'Q** *i e.*, the point of their intersection.

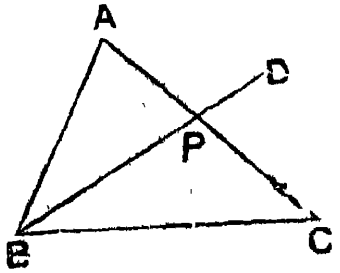
Hence each of the points **P**, **P'** satisfies the two conditions and is the point required.

Question. When is this problem impossible ?

From the above example we see that when a point satisfies *two* conditions, it must lie on each of the two loci which correspond to these conditions and must therefore be *the point common to these two loci*, i.e., *the point of their intersection.*

If in any case the data of the problem are such that the resulting loci *do not intersect*, then of course the problem is impossible. Thus if in Ex. 1, the radius of the circle be less than the distance of **O** from **X**, the solution would be impossible.

Ex. 2. Find a point (**P**) in a side (**AC**) of a triangle (**ABC**) equidistant from the other two sides (**AB**, **BC**).

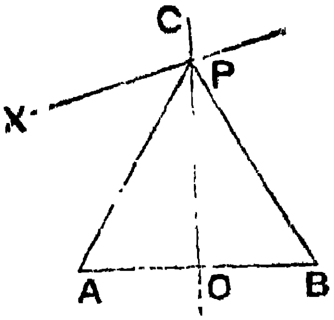


The locus of a point equidistant from **AB** and **BC** is the bisector **BD** of the ∠**ABC**. Hence the locus of **P** is **BD**.

Also the locus of **P** is **AC**.

∴ **P**, the point where the two loci **BD** and **AC** *intersect*, is the point required.

Ex. 3. Construct an isosceles triangle on a given base (**AB**), and having its vertex (**P**) on a given line (**X**).



Since the triangle is isosceles, its vertex **P** must be equidistant from **A** and **B**.

Now the locus of points equidistant from **A** and **B** is **OC**, the perpendicular bisector of **AB**.

Hence the locus of **P** is **OC**.

Also the locus of **P** is **X**.

∴ **P**, the vertex of the required △, is the point of intersection of **OC** and **X**.

Join **AP**, **BP**.

Thus **APB** is the triangle required, for **AP**=**BP** and **P** is in **X**.

Question. When is the solution impossible ?

EXERCISES XLI.

1. **A** and **B** are points 4·6 cm. apart. Find a point which is 3 cm. from **A** and 2·4 cm. from **B**. How many such points are there ?

2. Find a point equidistant from three given points, not in the same straight line.

3. Find a point in a given straight line equidistant from two given points. When is the problem impossible ?

4. Find a point at a given distance from a given point and a given straight line. When is the problem impossible ?

5. Find a point at a given distance from a given point and equidistant from the extremities of a given straight line. How many such points are there ?

6. Find a point which is 1 inch from a given point and 2 inches from a given straight line. How many such points do you find? When does the construction fail?

7. Find a point equidistant from the arms of an angle and is at a given distance from a fixed point in one of them.

8. **ABC** is a circle of which **AB** is a diameter and **AC** a chord. Find a point on the circle equidistant from **AB** and **AC**. How many such points are there?

9. Two straight lines **AB**, **CD** intersect at **O**. Find a point which is 3 cm. from **AB** and 4 cm. from **CD**. How many such points are there?

10. Find a point which is at a given distance from a given point and is equidistant (i) from two intersecting straight lines and (ii) from two parallel straight lines.

When are there two such points and when only one? When is the problem impossible?

11. Find a point equidistant from the two sides of a triangle as well as from the extremities of the base. How many such points can be found?

12. Find a point equidistant from the angular points of a triangle.

13. Find a point equidistant from the sides of a triangle.

14. Find a point equidistant from one side of a triangle and the other two sides produced.

15. **ABCD** is a trapezium, **AB** being parallel to **DC**. Find a point in **AC** which is equidistant from **AD**, **BC**.

16. Find a point which is at a given distance from either arm of a given angle.

17. Draw a circle which shall pass through two given points and have its centre in a given straight line. When does the construction fail?

18. Draw a circle of given radius that shall pass through two given points. How many solutions are there?

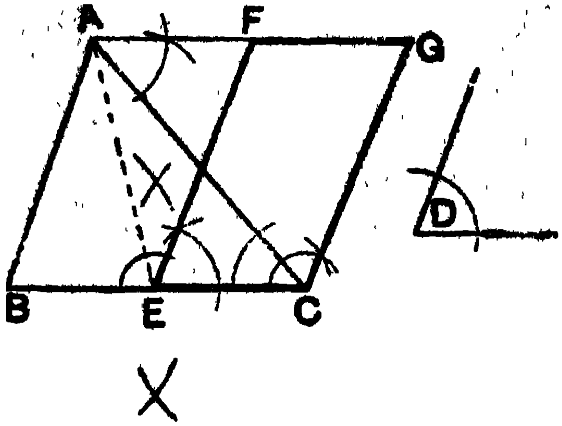
19. On a given base, construct a triangle of given altitude having its vertex on a given circle. How many such triangles can be found? When is the problem impossible?

20. Having given the base and the median that bisects it, construct a triangle whose vertex is equidistant from two intersecting straight lines. When is the problem impossible?

21. Construct a right-angled triangle, having given the hypotenuse and the perpendicular to it from the right angle. How many solutions are there?

CONSTRUCTIONS

Problem 1. To construct a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle.



Let ABC be the given triangle and D the given angle.

To construct a parallelogram equal in area to the △ABC *and having an angle equal to the* ∠D.

Construction. Bisect BC at E.

At E, make the ∠CEF = the ∠D.

Through C, draw CG ∥ to EF and through A, draw AFG ∥ to BC, meeting EF, CG in F and G.

Then ECGF is the required parallelogram.

Proof. Join AE.

Because the △^{s} ABE, AEC are on equal bases and have the same altitude,

∴ the △ABE = the △AEC; [II. 3, *Cor.* 1.

∴ the △ABC = 2△AEC.

And the parm. EG and the △AEC are on the same base and have the same altitude;

∴ the parm. EG = 2△AEC. [II. 3, *Cor.* 2.

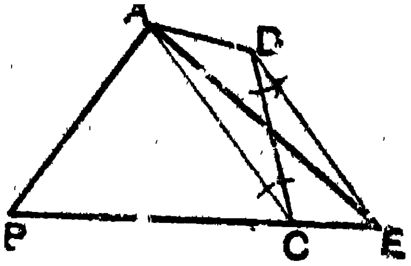
∴ the parm. EG = the △ABC.

And by construction it has the ∠CEF = the ∠D.

Hence EG is the required parallelogram.

Note. To construct a **rectangle** equal to the triangle **ABC**, erect a perp. EF at E.

Problem 2. To construct a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

To construct a triangle equal in area to ABCD.

[**Analysis.** Suppose ABE to be the triangle required. Join AC. By supposition the △ ABE = the quadl. ABCD. Take away the common △ ABC; thus the △ ACD = the △ ACE. Hence if DE be joined, then

DE is parallel to AC.

We have thus the following construction.]

Construction. Join AC.

Through D, draw DE ‖ to AC, meeting BC produced in E.

Join AE.

Then the △ ABE is the triangle required.

Proof. Because the △^{s} ACE, ACD are on the same base and of the same altitude,

∴ the △ ACE = the △ ACD.

To each add the △ ABC;

∴ the △ ABE = the quadrilateral ABCD.

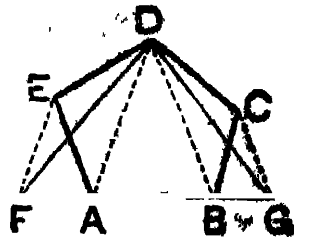
Note 1. Here the quadrilateral **ABCD** is reduced to the equivalent △**ABE**.

In a similar way, a pentagon may be reduced first to an equal quadrilateral and then to an equivalent triangle; a hexagon may be successively reduced to an equivalent pentagon, quadrilateral and triangle; and so on for any polygon. Hence

To construct a triangle equal in area to a given rectilineal figure,

we have to reduce the given rectilineal figure successively to equal figures of sides one less, until the proposed figure is finally reduced to an equivalent triangle.

Obs. The following is a convenient construction for the pentagon **ABCDE** :—Join **AD**, **BD**. Draw **EF** ∥ to **AD**, and **CG** ∥ to **BD**, meeting **AB** produced in **F** and **G** respectively. Join **DF**, **DG**.



Then the △ **DFG**=the pentagon **ABCDE**.

Note 2. The area of a rectilineal figure may be found—

(i) By reducing the given figure to an equivalent triangle as we have shewn above ; and

(ii) By dividing up the given figure into triangles.

We shall illustrate the methods by finding the area of a trapezium.

Let **ABCD** be the given trapezium, in which **AB** is ∥ to **CD**. Let **AB**$=a$, and **CD**$=b$; also let h be the height of the trapezium.

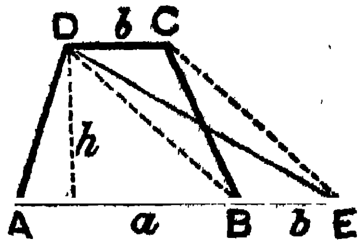
(i) Draw the diagonal **BD**.

Through **C**, draw **CE** ∥ to **DB**, meeting **AB** produced in **E**. Join **DE**.

Then the △**AED**=the trapezium **ABCD**.

Since **BECD** is a parallelogram,

∴ **CD**=**BE**$=b$.

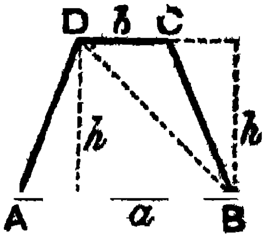


Hence the trapezium **ABCD**=the △ **AED**$=\frac{1}{2}h.$**AE**$=\frac{1}{2}h(a+b)$.

(ii) Draw the diagonal **BD**.

Then the trap. **ABCD**=the △ **ABD**+the △ **BCD**

$$=\frac{1}{2}h.a+\frac{1}{2}h.b=\frac{1}{2}h(a+b).$$



Thus we have the

THEOREM. **The area of a trapezium is equal to half the product of the height into the sum of the two parallel sides.**

Note 3. The above Theorem is very useful in finding the area of a polygon.

Ex. To find the area of the rectilineal figure ABCDEF.

Join **AD**. Draw perps. from **B, C, E, F** to **AD**.

$Ab=60$, $bf=75$, $fc=45$,

$ce=45$, $eD=30$. Also $Bb=90$, $Cc=105$, $Ee=75$, $Ff=135$.

$\Delta\ AbB=\frac{1}{2}\times60\times90=2700$;

trap. $Bc=\frac{1}{2}\times120(90+105)=11700$

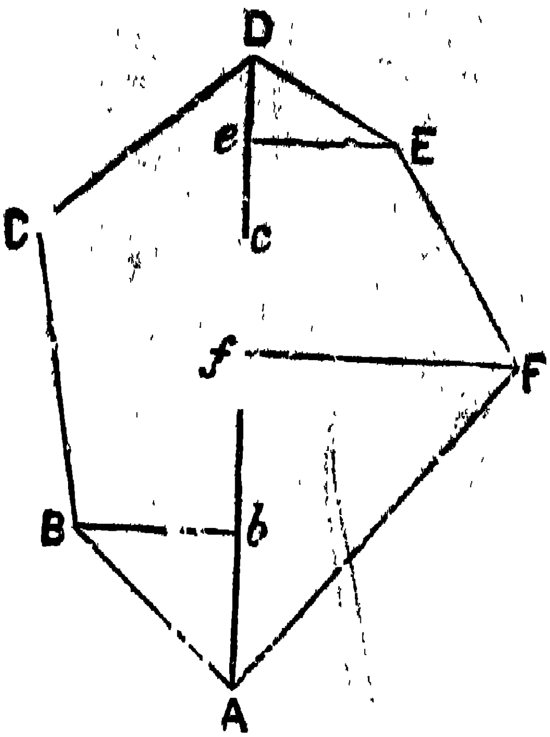
$\Delta\ CcD=\frac{1}{2}\times105\times75=3937\frac{1}{2}$;

$\Delta\ DeE=\frac{1}{2}\times30\times75=1125$;

trap. $eF=\frac{1}{2}\times90(75+135)=9450$:

$\Delta\ AfF=\frac{1}{2}\times135\times135=9112\frac{1}{2}$.

$\therefore$ area of fig. **ABCDEF** = sum of these areas = 38025 sq. units.



The method illustrated above, in which a polygon is divided up into right-angled triangles and trapeziums, is used by land-surveyors to measure lands. They note down their measurements in a book called **the field-book**, from which they make a sketch of the field and calculate its area as we have done above. The diagonal **AD** is called the **base-line** and the perpendiculars **B***b*, **C***c*, ...to the base-line are called **offsets**.

The measurements in a field-book, which are read *upwards*, are shewn thus :—

	To D	
	255	
	225	75E
C105	180	
	135	135F
B90	60	
	From A	

The numbers in the middle column are the distances of *b*, *f*, *c*, *e* and **D**, *all measured from* **A** and those in the other columns are the lengths of the offsets, right and left of the base-line.

Thus $Ab=60$; $Af=135$ and $\therefore$ $bf=135-60=75$; &c.

EXERCISES XLII.

[*Theoretical.*]

1. Construct a triangle equal to a given parallelogram and having an angle equal to a given angle.

2. Construct a parallelogram equal to a given trapezium.

3. On the same base as a parallelogram, construct a triangle equal to the parallelogram and having an angle equal to a given angle.

4. **Construct a parallelogram equal in area to a given rectilineal figure and having an angle equal to a given angle.** [Euc. I. 45.]

[Construct a triangle equal to the given rectilineal figure, Prob 2. *Note.*]

5. **ABCD** is a given quadrilateral; construct on **AB** a triangle of equal area having the angle **A** in common with the quadrilateral.

6. On a given base, construct a triangle equal in area to a given triangle and having its vertex in a given straight line.

7. On the longest side of a triangle, construct a rhombus equal in area to the triangle.

8. **On a given straight line, construct a triangle equal in area to a given triangle and having an angle equal to a given angle.**

Let **X** be the given straight line, **Z** the given angle, and **ABC** the given triangle.

Let **AB** be greater than **X**.

[**Analysis.** Suppose **APQ** to be the required triangle, having the ∠**PAQ**=the ∠**Z**.

Draw **QD** parl. to **AB**, meeting **AC** produced in **D**. Join **DP**.

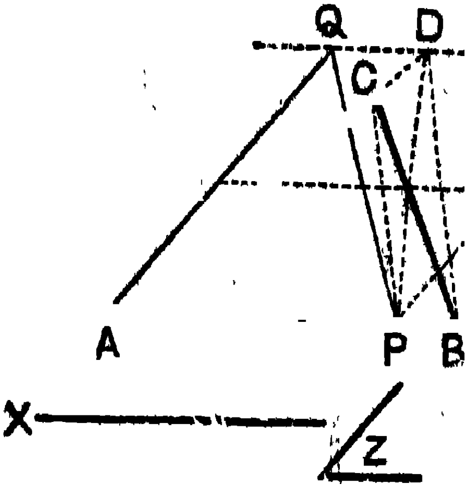
Then the △**APQ**=the △**APD**. Join **CP**.

Now the △**APD**=the △**APQ**=the △**ABC** by supposition; take away the △**APC**; ∴ the △**CPD**=the △**CPB**. Thus if **BD** be joined, then

BD is parallel to CP.

Hence the construction.]

Construction. Make **AP=X.** Join **CP** and through **B**, draw **BD** ∥ to **CP**, meeting **AC** produced at **D**. Make the ∠**PAQ**=the ∠**Z**. Through **D**, draw **DQ** parl. to **AB** to meet **AQ** in **Q**. Join **QP**.



Then **APQ** is the required Δ.

Proof. Join **DP**. Because the Δs **CPB**, **CPD** are on the same base **CP** and have the same alt.; they are equal in area.

To each add the Δ**APC** ; ∴ the Δ**ABC**=the Δ**APD**.

But the Δs **APD**, **APQ**, being on the same base **AP** and having the same alt., are equal.

∴ the Δ**APQ**=the Δ**ABC**

and it has the ∠**PAQ**=the ∠**Z**.

9. Construct a triangle equal in area to the sum of the areas of two given triangles.

10. Construct a triangle equal in area to the difference of the areas of two given triangles.

11. **On a given straight line, construct a parallelogram equal in area to a given triangle and having an angle equal to a given angle.** [Euc. I. 44.]

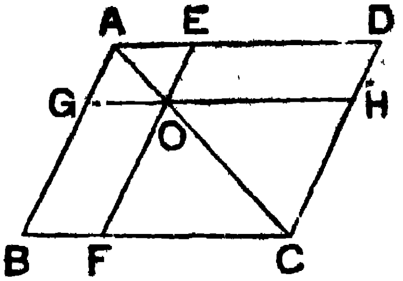
[As in Exercise 8, if **AP** be the given straight line, construct a Δ**APQ**= the given Δ and having the ∠**PAQ**=the given angle. Through the middle pt. of **AQ**, draw a parl. to **AP** and through **P**, draw a parl. to **AQ**.]

12. Bisect a triangle by a straight line drawn through an angular point.

13. Divide a triangle into any number of equal parts by straight lines drawn through an angular point. [(XXXVII.), 2.]

APPENDIX II

1. THEOREM. *If through any point* O *in the diagonal* AC *of a parallelogram* ABCD, EOF, GOH *are drawn parallel to* AB *and* AD *respectively, prove that the parallelogram* OB *is equal to the parallelogram* OD.



By I. 24, $\triangle ABC = \triangle ADC$,

$\triangle AGO = \triangle AEO$,

$\triangle OFC = \triangle OHC$.

$\therefore \triangle ABC - (\triangle AGO + \triangle OFC)$

$= \triangle ADC - (\triangle AEO + \triangle OHC)$;

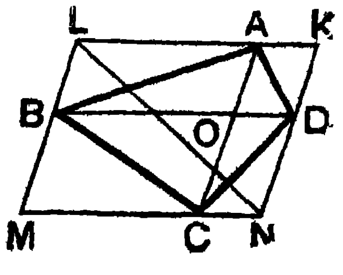
$\therefore$ the parm. OB = the parm. OD.

Note. This Theorem is the same as Euc. I. 43, which is enunciated thus :—

"The complements of the parallelograms which are about the diagonal of any parallelogram are equal."

The parms. OB, OD are called '*complements*,' because they with the parms. GE, and FH, which are about the diagonal AC, make up the whole parm. ABCD

2 THEOREM. *A quadrilateral is equal in area to a triangle having two of its sides equal to the diagonals of the quadrilateral and the included angle equal to either of the angles between the diagonals.*



Let ABCD be a quadl. and AC, BD its diagonals, which intersect at O.

To prove that ABCD is equal to a triangle whose sides are equal to AC, BD, and include an angle equal to $\angle AOD$.

Construction. Through A, B, C, D, draw parls. to the diagonals, to form the parm. KLMN.

Join LN.

Proof. Since KC, LD, and OK are parms.,

$\therefore$ KN = AC, KL = BD and the $\angle K$ = the $\angle AOD$.

Thus KLN is a triangle, whose sides are equal to the diagonals and include the ∠ K equal to an angle between the diagonals.

Now $\triangle$ ACD $= \frac{1}{2}$ the parm. AN $\}$ being on the same base and
and $\triangle$ ACB $= \frac{1}{2}$ the parm. AM $\}$ having the same altitude.

$\therefore \triangle ACD + \triangle ACB = \frac{1}{2}$ the parm. AN $+ \frac{1}{2}$ the parm. AM.

Hence the quadl. ABCD $= \frac{1}{2}$ the parm. KLMN $= \triangle$ KLN.

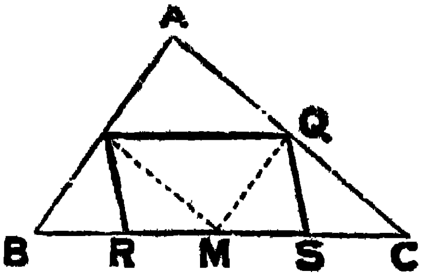
Cor. *If the diagonals intersect at right angles, then the area of the quadrilateral = half the rectangle contained by the diagonals.*

For in this case, the area of $\triangle$ KLN, (which is equal to the quadl. ABCD) $= \frac{1}{2}$KN.KL $= \frac{1}{2}$AC.BD.

3. THEOREM. *The straight line joining the middle points of the sides of a triangle and any pair of parallels drawn through these points to meet the base form with the base a parallelogram equal to half the triangle.*

Let ABC be a triangle and let P, Q be the middle points of the sides AB, AC, through which any two parallels PR and QS are drawn to meet the base BC in R and S.

To prove that the parallelogram PQSR is half the triangle ABC.



Construction. Bisect BC at M; and join PM, QM.

Proof. Then PQ is parl. to BC; PM to AC and QM to AB. Thus PQSR, APMQ, PBMQ and PMCQ are parms.

Now the $\triangle$ PQM is equal to each of the $\triangle$ˢ APQ, BPM and CQM [I. 24]; and these four $\triangle$ˢ make up the $\triangle$ ABC; hence the $\triangle$ PQM is $\frac{1}{4} \triangle$ ABC.

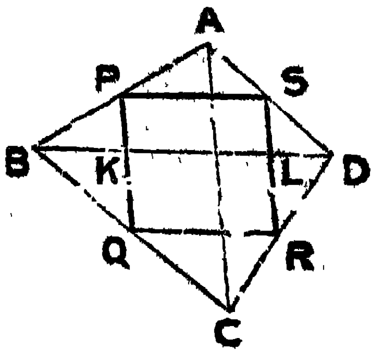
But the parm. PQSR and the $\triangle$ PQM are on the same base PQ, and have the same altitude;

$\therefore$ parm. PQSR $= 2 \triangle$ PQM;

$\therefore$ parm. PQSR $= \frac{1}{2} \triangle$ ABC.

4. THEOREM. *The straight lines that join the middle points of the adjacent sides of a quadrilateral form a parallelogram equal to half the quadrilateral*

Let **ABCD** be a quadl., whose diagonals are **AC**, **BD**.



Let **P**, **Q**, **R**, **S** be the mid. pts. of **AB**, **BC**, **CD**, **DA**.

To prove that **PQRS** is a parm. equal in area to half the quadl. **ABCD**.

Proof. ∵ **PQ**, **SR** are each ∥ to **AC** ;

∴ they are parallel.

Similarly, **PS** is parallel to **QR**.

Hence **PQRS** is a parm.

Again, let **PQ**, **RS** cut **BD**, in **K**, **L**.

Then parm. **PL** $= \frac{1}{2} \triangle$ **ABD**, and parm. **QL** $= \frac{1}{2} \triangle$ **CBD**.

∴ by addition, parm. **PQRS** $= \frac{1}{2}$ sum of $\triangle^s$ **ABD**, **CBD**

$= \frac{1}{2}$ the quadl. **ABCD**.

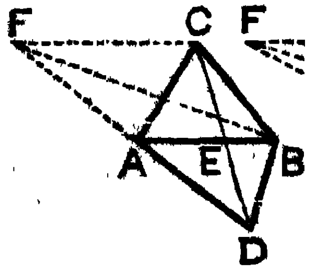
Note. Suppose the quadrilateral **ABCD** to consist of the two triangles **ABD**, **CBD**. Then this Theorem proves that the parm. **PQRS** is half the *sum* of the $\triangle$s **ABD**, **CBD**, if they are on the same base **BD** but on opposite side of it.

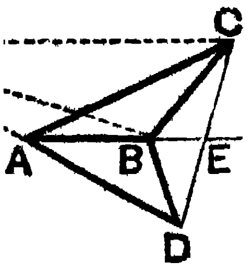
But *if the triangles are on the same base* **BD** *and on the same side of it, the parm.* **PQRS**, *shall be half the difference of the* $\triangle^s$ **ABD**, **CBD**.

Prove this case.

[Produce **PQ**, **RS** to meet the base in **K** and **L**.]

5. THEOREM. *Two equal triangles are on the same base and on opposite sides of it ; prove that the join of their vertices is bisected by the base, or the base produced.*





Let **ABC**, **ABD** be two equal $\triangle^s$ on the same base **AB** and on opposite sides of it.

To prove that CD, joining their vertices, is bisected at E by AB, or AB produced.

Construction. Draw CF ∥ to AB, meeting DA produced at F. Join BF.

Proof. Because the △ˢ ABF, ABC are on the same base and have the same altitude ;

∴ the △ ABF = the △ ABC

= the △ ABD (*Hyp.*) ;

∴ AF = AD.

Hence AE, being ∥ to CF through the middle pt. A of FD, bisects CD at E.

Otherwise :—Draw CP, DQ perp. to AB. Then ∵ CP = DQ, the △ˢ CPE, DQE are equal in all respects, and consequently CE = ED.

6. THEOREM. *Two triangles are on the same base and on opposite sides of it ; if the join of their vertices is bisected by the base, or the base produced, prove that the triangles are equal in area.* [Converse of (5)].

7. THEOREM. *Any parallel to a side of a triangle is bisected by the corresponding median.*

Let ABC be a triangle ; and AD the median to BC.

Let XY be any parl. to BC.

To prove that XY is bisected by AD.

Proof. Join DX, DY.

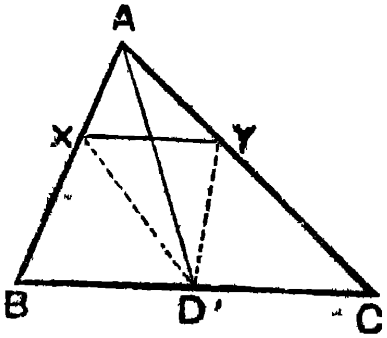
Because BD = CD,

∴ the △ ABD = the △ ACD,

and the △ BDX = the △ CDY.

By subtraction,

the △ AXD = the △ AYD.



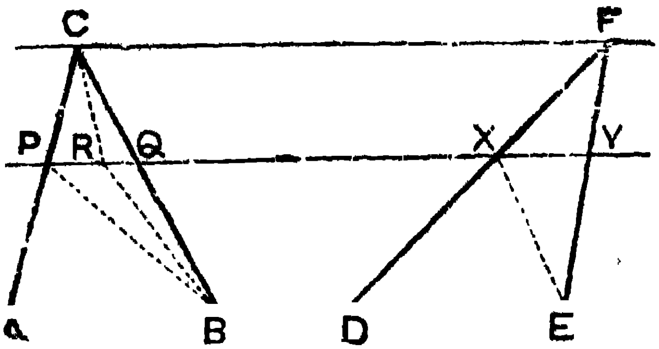
Hence XY, which joins the vertices of the two equal △ˢ AXD, AYD, is bisected by the base AD, by (5).

Obs. The Theorem holds likewise when the parallel XY cuts BA and CA produced. Prove this case.

Ex. 1. **AM** is a median of a triangle **ABC**; through any pt. **O** in it, **POQ** is drawn parl. to **BC**; to prove that

the trap. **POMB** = the trap. **QOMC**.

8. THEOREM. *If two triangles are on equal bases and between the same parallels, the intercepts made by the sides of the triangles on any parallel to the line of bases are equal.*



Let ABC, DEF be two $\triangle^s$ on equal bases AB, DE and between the same parallels AE, CF.

Let PQXY be any parallel to AE.

To prove that the intercept PQ = the intercept XY.

Proof. If PQ, XY are not equal, let PQ be greater than XY.

From PQ, cut off PR = XY.

Join CR, RB, PB, XE.

Because AB = DE,

∴ the $\triangle$ABC = the $\triangle$DEF and the $\triangle$ABP = the $\triangle$DEX;

∴ by subtraction, the $\triangle$CBP = the $\triangle$FEX.

Again ∵ PR = XY,

∴ the $\triangle$PRC = the $\triangle$XYF and the $\triangle$PRB = the $\triangle$XYE;

∴ by addition, the fig. CRBP = the $\triangle$FEX.

Hence the fig. CRBP = the $\triangle$CBP, a part equal to the whole, which is absurd.

∴ PQ is not greater than XY, *i.e.*, PQ = XY.

Obs. The Theorem likewise holds in the case when the parallel **PQXY** cuts **AC**, **BC** produced and **DF**, **EF** produced. Prove this case.

Note. It will be seen that (7) is a particular case of this general Theorem.

Cor. *The straight line drawn parallel to the parallel sides of a trapezium through the point of intersection of the diagonals is bisected at that point.*

MISCELLANEOUS EXERCISES.

1. A trapezium is bisected by the join of the middle points of the parallel sides.

2. **AB**, **CD** are two parallel straight lines; **P**, **Q** are points that move along **AB** and **CD** respectively, such that **AP** is always equal to **CQ**. Find the locus of the middle point of **PQ**.

3. On one side of a square, describe an isosceles triangle equal in area to the square.

4. The square on a line is four times the square on its half.

5. The square on a straight line is equal to 9 times the square on its third part.

6. If two equal parms. **ABCD**, **ABPQ** are on the same base **AB** and on opposite sides of it, prove that **DQ** is bisected by **AB**; and *conversely*, if **DQ** is bisected by **AB**, then parm. **ABCD** = parm. **ABPQ**.

7. The locus of the vertex of a triangle on a fixed base and of constant area, is a pair of straight lines parallel to the base.

8. Find the locus of the intersection of the diagonals of a parallelogram standing on a given base and having a constant area.

9. Produce a given straight line, so that the rectangle contained by the whole line thus produced and the given line may be equal to a given square.

10. **ABC** is a triangle; **AB** is produced to **D**, so that **BD**=**AB** and **D** is joined to the middle point **E** of **AC**; if **DE** meets **BC** in **H**, shew that the Δs **BHD**, **EHC** are equal in area.

11. **ABC** is a triangle; **D**, **E** are the middle points of **AB**, **AC**; **F**, **G** are the points of trisection of **BC**; **DF**, **EG** are produced to meet at **H**; prove that the Δ**FGH** = $\frac{1}{3}$ Δ**ABC**.

[*Hint.* **AFHG** is a parallelogram.]

12. **ABCD** is a parm.; through any point **O** in **AC**, parallels are drawn to **AD** and **AB**, meeting **AB**, **CD** in **E**, **F** and **AD**, **BC** in **G**, **H**; shew that the Δ**ABC** = the Δ**AFH** + the Δ**AEG**.

13. **ABC** is a triangle; **PQR** is a straight line parallel and equal to **BC** meeting **AB**, **AC** in **P** and **Q**; prove that $\triangle$**BQP**$=\triangle$**AQR**.

14. Through any point **D** in a side **AB** (or **AB** produced) of a triangle **ABC**, draw a straight line to meet **BC** produced (or **BC**) in **E**, such that $\triangle$**BDE**$=\triangle$**ABC**.

15. If a, b denote the sides of a right-angled triangle containing the right-angle and p the perpendicular from the right angle to the hypotenuse, then $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

16. The sum of the perpendiculars drawn from any point within an equilateral triangle to the sides is constant.

17. **P** is any point in the diagonal **AC** of a parallelogram **ABCD**; prove that the $\triangle$**APB**+the $\triangle$**CPD**$=\frac{1}{2}$ of the parallelogram.

18. **PQRS** is a parallelogram. Find the locus of the remaining vertices of the parallelogram equal in area to **PQRS**, having **PR** as diagonal.

19. **ABCD** is a quadrilateral. Find the locus of **P** which moves so that the quadrilateral **ABCP** is always equal in area to **ABCD**.

20. **The sum of the distances of any point within a regular polygon from the sides is constant.**

21. Construct a rhombus equal in area to a given parallelogram.

22. Construct a triangle equal to a given triangle and having two of its sides of given lengths. When is this problem impossible?

23. **APB**, **ADQ** are two straight lines such that the $\triangle$**APQ**$=$ the $\triangle$**ABD**; if the parallelogram **ABCD** is completed and **BQ** joined cutting **CD** in **R**, shew that **AP**$=$**CR**.

24. **ABC** is a triangle; on **BC** a parm. **BDEC** is constructed, and parms. whose adjacent sides are **AB**, **AD** and **AC**, **AE** respectively, are completed; prove that the first parm. is the sum or difference of the last two.

[*Hint.* Through **A**, draw a parl. to **BD** or **CE** to cut **BC** in **K** and **DE** in **L**. Then $\triangle$**ABD**$=\frac{1}{2}$ parm. **KD** and $\triangle$**ACE**$=\frac{1}{2}$ parm. **KE**].

25. **Two sides of a triangle being given, prove that its area is a maximum when the included angle is a right angle.**

26. Of all parallelograms which can be formed with given adjacent sides, the rectangle has the greatest area.

27. In an acute-angled triangle **ABC**, if **O** is the point where the perpendiculars from the vertices to the opposite sides meet, prove that

$$OA.BC+OB.CA+OC.AB=4\triangle ABC.$$

28. Divide a given straight line into two parts such that the square on one part may be equal to three times the square on the other part.

29. **D**, **E**, **F** are the mid. points of **BC**, **CA**, **AB** of a triangle

ABC and O is the point of intersection of the medians; prove that the triangle BOD is one-sixth of the $\triangle ABC$.

30. Find a point O within a triangle ABC such that
the $\triangle OAB =$ the $\triangle OBC =$ the $\triangle OCA$.

31. Find a point O within a quadrilateral ABCD such that
the $\triangle AOB =$ the $\triangle BOC =$ the $\triangle COD$.

32. ABCD is a rectangle; E is any point in BC and F in CD; shew that the rect. $ABCD = 2 \triangle AEF +$ the rect. BE, DF.

[*Hint.* Draw EOG, FOH ||s to AB, AD intersecting at O. Join OA, and prove the $\triangle AEF$ is half the rects. OB, OC and OD.]

33. ABC is a triangle; from A and C, ADF, CFE are drawn perpendicular respectively to the internal and external bisectors of $\angle B$ prove that the rect. BDFE = the $\triangle ABC$.

[*Hint.* Draw CGH || to AB cutting AF and BE in G and H. Then rect. BDFE and $\triangle ABC$ are each half of the parm. AGHB].

34. **On the sides AB, AC of a triangle ABC, parallelograms ABDE, ACFG are constructed outside the triangle; if DE, FG intersect at H, prove that the parallelogram whose adjacent sides are BC and a straight line parallel and equal to AH is equal to the sum of the parallelograms ABDE, ACFG.**

35. In Ex. 34, prove that the parm. whose adjacent sides are BC and CK (where CK is a parallel to AH meeting FH in K) is equal to the sum of the parms. ABDE, ACFG

36. The perimeter of an isosceles triangle is greater than that of an equal rectangle of the same altitude

37. O is a point outside the parm. ABCD; prove that the sum or difference of the triangles OAB, OCD is half the parallelogram.

38. ABCD is a triangle and D is a movable point in the base BC; from D, DE, DF are drawn parallel to AB and AC respectively, meeting AC, AB in E, F. Find the locus of the middle point of EF.

39. If a triangle and a square be of equal area, then the perimeter of the triangle is greater than that of the square.

40. In a right-angled triangle, the equilateral triangle on the hypotenuse is equal to the sum of the equilateral triangles on the sides containing the right angle. [Compare II. 5.]

41. On the sides of any triangle, squares are described, all external to the triangle; prove that the three triangles formed by joining the adjacent corners of these squares are equal in area to one another and to the original triangle.

42. **ABCD is a parm. and O is any point outside it and without the angle made by producing BA, DA; prove that**

$$\triangle OAC = \triangle OAB + \triangle OAD.$$

43. **In Ex. 42, if O is within the parm. or within the angle made by producing BA, DA, then $\triangle OAC = \triangle OAB \sim \triangle OAD$.**

BOOK III

THE CIRCLE

BOOK III

The Circle

DEFINITIONS

The following definition of a circle in which the idea of 'locus' is involved, is often very useful.

1. A **circle** is a plane figure bounded by a line traced out by a point which moves so as to be always at a constant distance from a fixed point.

The line traced out by the point is called the **circumference**, the fixed point is called the **centre** and the constant distance is called the **radius**.

Hence

(i) All radii of the same circle are equal.

(ii) A point is within, on or without, a circle according as its distance from the centre <, = or > the radius.

(iii) Circles of equal radii are equal.

For the radii being equal, the circumference of each circle encloses the *same* space.

Note. The word "circle" strictly denotes the *space* enclosed by the circumference. The term, however, is often used to denote the circumference itself, if no confusion is likely to arise.

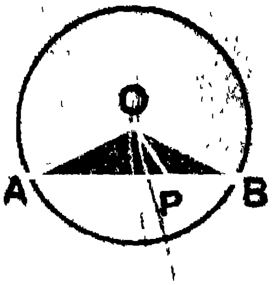
2. It also appears from the above definition that a circle is a *closed* figure and therefore if a straight line cuts a circle at *one point*, it will, if produced, cut the circle at a *second point*. Thus

A straight line cuts a circle at two points.

3. A **chord** of a circle is the straight line joining any two points on the circumference.

From this definition it is easy to see that

A chord of a circle falls within the circle [Euc. III. 2].



[Let AB be a chord of a ⊙ of centre O,
and P any pt. in AB.
Join OA, OB, OP.
Then the ∠OAB = the ∠OBA
but the ∠OPA > the ∠OBA;
∴ the ∠OPA > the ∠OAB;
∴ OA > OP; that is, OP < the radius;
thus P is within the ⊙.

And ∵ P is any pt., *every* pt. in AB between A and B is within the ⊙. Hence the chord AB falls within the ⊙.]

4. **Concentric circles** are those which have the same centre.

Hence two concentric circles of equal radii must coincide. But if they have unequal radii, one must fall entirely within the other. Thus

Two concentric circles of unequal radii cannot cut one another.

5. The **line of centres** of two circles is the straight line drawn through their centres.

SYMMETRY

6. **Symmetry about a line.** A figure is said to be **symmetrical about a line** in it, when the figure is folded about that line, the two parts on each side of the line coincide with one another.

The line is called an **axis of symmetry** of the figure. Thus a circle is symmetrical about a diameter, a rhombus is symmetrical about a diagonal, &c.

Question. How many axes of symmetry have a *square*, a *rectangle*, a *parallelogram*, a *rhombus* and a *circle*?

If a figure is symmetrical about a line, then either half of the figure is called an **image** or **reflection** of the other half in that line.

If from P, PC is drawn perpendicular to AB and produced to P′, so that CP′ = CP, then AB is the axis of symmetry of the line PP′, and P′ is called the **image** of P in AB.

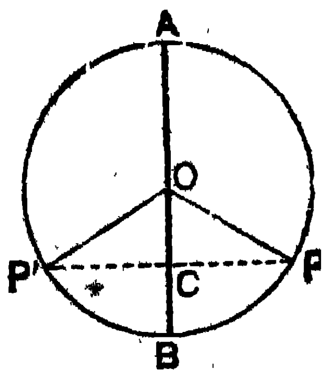
In fact in a symmetrical figure, *every point* in either half has an *image* in the other half.

A point and its image are called **symmetrically opposite** points with regard to the axis.

7. From the definition, it is almost self-evident that a circle is symmetrical about a diameter. The following, however, is a formal proof of this

Theorem. *A circle is symmetrical about any diameter.*

Let there be a circle with centre O; and let AB be any diameter.



Then the circle is symmetrical about AB.

Proof. Let P be any point on the arc APB. Join OP.

Fold the circle about AB, and let the point P in its new position fall at P′

Then OP coincides with OP′.

Thus OP′ is a radius of the circle.

∴ P′ is a point on the arc AP′B.

Since P is any point, it follows that *every* point on the arc APB falls on the arc AP′B.

∴ the two semi-circles APB, AP′B coincide.

Hence the circle is symmetrical about AB.

Cor. 1. *Two circles are symmetrical about their line of centres.*

For the portion of the line of centres intercepted by each ⊙ is a diameter of the ⊙, and the two diameters are in the same st. line.

Cor. 2. *A circle that passes through a given point shall also pass through the symmetrically opposite point with regard to any diameter.*

For if **P**, **P′** are symmetrically opp. points with regard to the diameter **AB**, then when the ⊙ is folded about **AB**, the semi-⊙s **APB**, **AP′B** coincide and **P** coincides with **P′**. Hence **P′** is on the circle.

Cor. 3. *A chord of a circle which joins two symmetrically opposite points with respect to a diameter is bisected at right angles by the diameter.*

Let **PP′** be the chord, cutting **AB** at **C**.

Fold the circle about **AB**. Then **P** coincides with **P′**, **CP** with **CP′** and the ∠**OCP** with the ∠**OCP′**.

Thus **CP** = **CP′**; and the ∠**OCP** = the ∠**OCP′**, which being adjacent angles are right angles.

Cor. 4. *If two circles intersect at one point, they shall also intersect at a second point viz., the symmetrically opposite point with regard to the line of centres.*

Let two circles intersect at **A**; and let **B** the symmetrically opp. pt. to **A** with regard to the line of centres. Then prove as in *Cor.* 2, that each of the circles passes through **B**.

Cor. 5. *The line of centres of two circles bisects their common chord at right angles.*

This follows at once from *Cor.* 3.

Question. Prove by symmetry that a diameter of a circle at right angles to a chord bisects the chord [III. 1.]

7. **Symmetry about a point.** **Two points** are said to be symmetrical about a point, if their join is bisected by the point. Thus the points P, P′ in the above figure are symmetrical about C.

A **figure** is said to be symmetrical about a point, if every line in the figure passing through the point is bisected by it. Thus a circle is symmetrical about its centre; a parallelogram is symmetrical about the intersection of the diagonals, and so on.

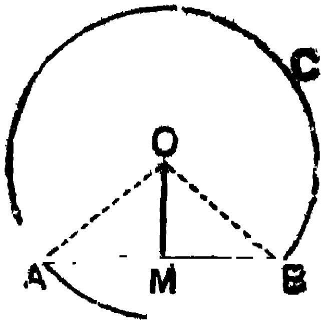
Question 1. Name some other rectilineal figures that are symmetrical about a point.

Question 2. If a quadrilateral is symmetrical about a point, prove that it is a parallelogram.

THEOREM 1. [Euc. III. 3.]

(i) A straight line drawn from the centre of a circle to bisect a chord which is not a diameter, is perpendicular to the chord.

(ii) Conversely, the perpendicular to a chord from the centre bisects the chord.



Let ABC be a circle of which the centre is O; and let AB be a chord which is not a diameter.

(i) Let OM bisect AB at M.

To prove that OM *is perpendicular to* AB.

Construction. Join OA, OB.

Proof. In the △ˢ AMO, BMO,

(i) AM = BM. *Hyp.*

(ii) OM is common,

and (iii) OA = OB, being radii of the circle;

∴ the ∠ AMO = the ∠ BMO; [I. 7.

and these being adjacent angles are right angles;

∴ OM is perpendicular to AB.

(ii) *Conversely*, let OM be perp. to the chord AB.

To prove that OM *bisects* AB.

Construction. Join OA, OB.

Proof. In the right-angled △ˢ AMO, BMO,

(i) the hypot. OA = the hypot. OB,

and (ii) the side OM is common;

∴ AM = BM, [I. 20.

that is, OM bisects AB at M.

Q.E.D.

Cor. 1. *The perpendicular bisector of any chord of a circle passes through the centre.*

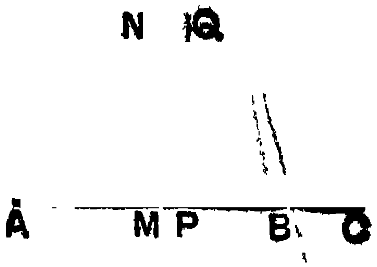
For at **M** *only one* perpendicular can be drawn to **AB**.

Cor. 2. *A straight line cannot cut a circle at more than two points.*

If possible, let a st. line cut a ⊙ at three pts. **A**, **B** and **C**.

Let **MN**, **PQ** be the perpendicular bisectors of **AB**, **AC** respectively.

Since **AB**, **AC** are both chords of the ⊙, ∴ **MN**, **PQ** both pass through the centre; and because they are parl., ∴ they do not intersect.



Hence the same ⊙ has two centres, which is absurd.

EXERCISES XLIII.

[*Numerical and Graphical.*]

1. Draw a circle having a diameter 5 cm. in length and place in it a chord 2 cm. in length. Find the distance of the chord from the centre and verify your calculation by measurement.

2. Draw a circle of radius 1·7″ and in it place a chord at a distance of 1·5″ from the centre; find the length of the chord, and verify by mesurement.

3. In a circle a chord 8 cm. in length is at a distance of 7·5 cm. from the centre; find the radius of the circle.

4. **AB** is a diameter of a circle 10 cm. in length; it bisects a chord **PQ** whose distance from the centre is 3 cm.; find **AP**, **BP**.

5. The sum of the diameter and a chord of a circle is 18″; if the chord is at a distance of 6″ from the centre, find the diameter of the circle.

6. The difference between the diameter and a chord of a circle is 1″; if the distance of the chord from the centre is 2·5″, find the length of the chord.

[*Theoretical.*]

7. **The perpendicular bisectors of any two chords of a circle intersect at the centre.**

[This follows from *Cor.* 1.

8. **Find the centre of a given circle** [Euc. III. 1.]

[Draw any two chords and bisect them at right angles.]

9. **Two chords of a circle which do not both pass through the centre cannot bisect one another** [Euc. III. 4.]

10. **If two circles have one point in common, they cannot have the same centre** [Euc. III. 5 and 6.]

11. If the straight line joining the middle points of two chords of a circle is perpendicular to one of them, it is likewise perpendicular to the other.

12. **AB, AC** are equal chords of a circle; shew that they make equal angles with the diameter through **A**

13. **The line of centres of two circles bisects their common chord at right angles.**

[Let **AB** be the common chord of two circles whose centres are **K** and **L**. Bisect **AB** at **M** and join **KM**, **LM**. Then **KM** and **LM** are each perp. to **AB** at **M**. Hence **KML** is one st. line.]

14. If a straight line cut two concentric circles, the parts of it intercepted between the circumferences are equal.

15. Through a given point within a circle, draw the chord which is bisected at that point.

16. Through either point of intersection of two circles whose centres are **A** and **B**, a parallel is drawn to **AB**, terminated by the circumferences; prove that this line is double of **AB**.

17. **Of all straight lines that can be drawn through a point of intersection of two circles and terminated by the circumferences the greatest is that which is parallel to the line of centres.**

18. The join of the middle points of two parallel chords of a circle passes through the centre.

19. **The locus of the middle points of parallel chords of a circle is a diameter.**

20 **Find the locus of the centres of circles that pass through two given points.**

21. Draw a circle of given radius that shall pass through two given points. When is this problem impossible?

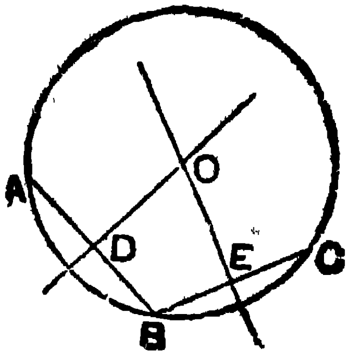
22. **Prove that all circles that pass through a fixed point and have their centres in a fixed straight line, pass also through a second fixed point.**

23. Two pairs of equal straight lines are drawn from a given point to a circle; prove that the chords joining the ends of the unequal straight lines intersect on the diameter through the given point.

24. From an external point two equal straight lines are drawn to cut a circle; prove that they are equidistant from the centre of the circle.

THEOREM 2.

There is one circle, and one only, which passes through three given points not in a straight line.



Let A, B, C be three given points not in a straight line.

To prove that *one circle, and one only, passes through* A, B *and* C.

Construction. Join AB, BC.

Let DO, EO be the perpendicular bisectors of AB and BC respectively.

Proof. Because AB, BC are not in the same st. line, DO and EO are not parl. and must therefore intersect at some point O.

Then because DO is the perp. bisector of AB,

∴ any point in DO is equidistant from A and B. [II. 7.
Similarly, any point in EO is equidistant from B and C.

∴ O the point common to DO and EO, is equidistant from A, B and C.

And because two st. lines can intersect at but one point,

∴ O is the only one point which is equidistant from A, B and C.

Hence the circle with centre O and radius OA is the *only circle* which passes through A, B and C.

Q.E.D.

Question. Explain why the pts. should not be in the same st. line

Cor. 1. *Circles which have three points in common coincide.*

For one circle, and one only, can be drawn through the three points.

Cor. 2. *Two circles cannot cut one another in more than two points* [Euc. III, 10].

For if they have a third point in common, they coincide.

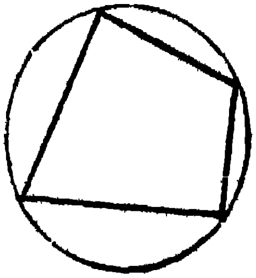
Cor. 3. *If from a point within a circle three equal straight lines can be drawn to the circumference, that point is the centre of the circle* [Euc. III. 9].

Let **O** be a pt. within a circle from which three equal st. lines **OA OB, OC** are drawn to the circumference.

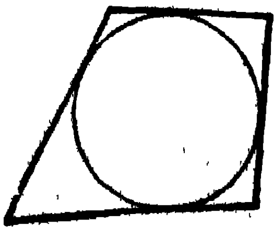
Since **OA=OB=OC** ; ∴ **O** is the centre of the circle passing through **A, B** and **C** ; and *one* circle and *one* only can pass through the three pts. **A, B** and **C** ; ∴ **O** is the centre of the circle.

Otherwise: Join **AB, BC.** Then ∵ **OAB** is an isosceles △, ∴ the perp. from **O** bisects **AB** ; that is, the perpendicular bisectors of **AB** passes through **O.** Similarly the perpendicular bisector of **BC** passes through **O.** Thus the perpendicular bisectors of two chords **AB** and **BC** intersect at **O** ; ∴ **O** is the centre [(XLIII), 7].

Def. 1. If all the angular points of a polygon are on the circumference of a circle, the polygon is said to be **inscribed** in the circle ; and the circle is said to be **circumscribed** about the polygon.



Def. 2. If a circle touches all the sides of a polygon, the circle is said to be **inscribed** in the polygon ; and the polygon is said to be **circumscribed** about the circle.



EXERCISES XLIV.

[*Graphical and Theoretical.*]

1. **AB**=4 *cm.* and **BC**=4·8 *cm*, and they include an angle of 72°; draw a circle passing through **A, B, C**, and measure its radius.

2. **ABC**, is a triangle whose sides are 2·4″, 1·8″ and 3″; describe a circle round the △**ABC** and find its radius. Verify by measurement.

3. Draw a circle to pass through the points (11, 0), (0, 9) and (−5, 0), and find its radius by measurement. Does the circle pass through the points (6, 9), (9, −4), (0, −6) and (−5, 2)?

[Use squared paper.]

4. The perpendicular bisectors of the sides of a triangle are concurrent.

5. **Circumscribe a circle about a given triangle.** [Euc. IV. 5]

6. **ABC** is an isosceles triangle having **AB**=**AC**; shew that the bisector of the angle **A** contains the centre of the circle circumscribed about the triangle **ABC**.

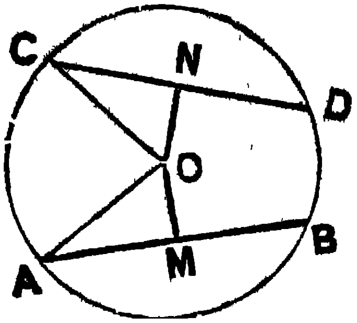
7. **ABCD** is a quadrilateral in which the angles **A** and **C** are acute; the perpendicular bisectors of **AB**, **AD** meet at **P** and those of **CB**, **CD** at **Q**; prove that **PQ** bisects **BD** at right angles.

8. **An arc of a circle being given, to complete the circle of which it is an arc.** [Euc. III. 25.]

THEOREM 3. [Euc. III. 14.]

(i) Equal chords of a circle are equidistant from the centre.

(ii) Conversely, chords of a circle which are equidistant from the centre are equal.



Let AB, CD be chords of a circle whose centre is O.

Let OM, ON be perpendiculars to AB and CD from O; so that OM, ON are the distances of AB and CD from O.

(i) Let AB = CD

To prove that AB *and* CD *are equidistant from* O.

Construction. Join OA, OC.

Proof. Because OM is perp. to the chord AB;

∴ OM bisects AB; [III. 1.

∴ $AM = \frac{1}{2}AB$.

Similarly $CN = \frac{1}{2}CD$.

But AB = CD; [*Hyp.*

∴ AM = CN.

Then in the right-angled △ˢ OAM, OCN,

(i) the hypot. OA = the hypot. OC.

and (ii) one side AM = one side CN; [*Proved.*

∴ OM = ON; [I. 20.

that is, AB and CD are equidistant from O.

(ii) *Conversely*, let OM = ON.

To prove that AB = CD.

Proof. As before, it may be proved that

$AM = \frac{1}{2}AB$, and $CN = \frac{1}{2}CD$.

Now in the right-angled $\triangle^s$ OAM, OCN,

(i) the hypot. OA = the hypot. OC, [*Hyp.*

and (ii) one side OM = one side ON ; [I. 20.

∴ AM = CN ;

∴ 2AM = 2CN ;

that is, AB = CD.

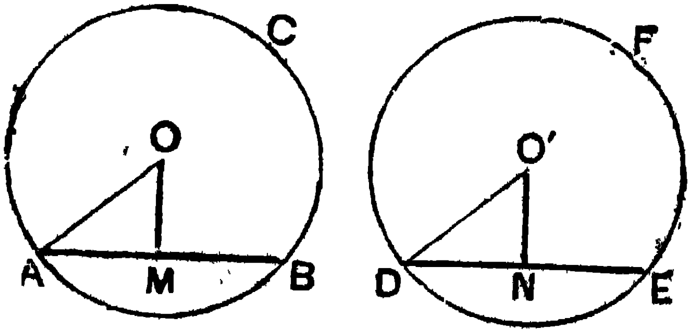
Q.E.D.

Note. The above Theorem is a particular case of the general Theorem given below.

In equal circles (or *in the same circle*),—

(i) *Equal chords are equidistant from the centres* ;

(ii) Conversely, *chords equidistant from the centres are equal.*



Let ABC, DEF be two equal circles, whose centres are O, O.′

Let OM, O′N be perpendiculars on the chords AB, DE, so that OM, O′N are their distances from the centres.

(i) Let AB = DE.

To prove that OM = O′N.

Construction. Join OA, O′D.

Proof. Because OM is perp. to the chord AB

$\therefore$ OM bisects AB ;

$\therefore$ AM $= \frac{1}{2}$AB.

Similarly DN $= \frac{1}{2}$DE.

But AB $=$ DE. [*Hyp.*

$\therefore$ AM $=$ DN.

Then in the right-angled $\triangle^s$ OAM, O′DN,

(i) the hypot. OA $=$ the hypot. O′D,

and (ii) one side AM $=$ one side DN ;

$\therefore$ OM $=$ O′N.

(ii) Let OM $=$ O′N.

To prove that AB $=$ DE.

Proof. As before, it may be proved that

AM $= \frac{1}{2}$AB and DN $= \frac{1}{2}$DE.

Then in the right-angled $\triangle^s$ OAM, O′DN,

(i) the hypot. OA $=$ the hypot. O′D,

and (ii) one side OM $=$ one side O′N ;

$\therefore$ AM $=$ DN.

$\therefore$ 2AM $=$ 2DN, that is, AB $=$ DE.

Q.E.D.

EXERCISES XLV.

[*Graphical and Theoretical*]

1. A chord 16″ in length is distant 6″ from the centre ; find the length of a chord distant 4″ from the centre, and the distance from the centre of a chord 12″ long.

2. Chords 8 cm. in length are placed in a circle whose radius is 5 cm. ; shew that the middle points of all these chords lie on a circle whose radius is 3 cm.

3. **The line joining the middle points of two equal chords of a circle is bisected by the perpendicular from the centre.**

4. Through a given point in a chord of a circle, draw a chord equal to it.

5. A diameter of a circle cuts two parallel chords in points equidistant from the centre; prove that the chords are equal.

6. Two parallel chords of a circle on opposite sides of the centre are 6 and 7 cm. and their mid-points are 4·6 cm. apart; find the radius of the circle. Verify by measurement.

7. Two parallel chords of a circle are 16 cm. and 30 cm. in length; find the distance between them, if the radius of the circle is 17 cm.

8. One of two parallel chords of a circle is 30 cm. and the line joining their mid points is 27 cm.; if the diameter of the circle is 50 cm., find the other chord and its distance from the centre.

9. **Any two equal chords of a circle make equal angles with the diameter through their point of intersection;** and *conversely*, **any two chords making equal angles with the diameter through their point of intersection are equal.**

[The point of intersection may be *within*, *on* or *without* the circle.]

10. Equal chords of a circle at the extremities of a diameter make equal angles with the diameter; and *conversely*, chords making equal angles with a diameter at its extremities are equal.

11. **Parallel chords of a circle at the extremities of a diameter are equal.**

12. If two equal chords of a circle intersect, either within or without it, the segments of one are equal respectively to the corresponding segments of the other.

13. If two chords of a circle intersect so that a segment of one is equal to the corresponding segment of the other, the chords are equal.

14. **Three equal chords of a circle cannot intersect at a point within it, unless that point is the centre.**

15. Parallel chords of a circle at the extremities of a diameter meet the circumference at the extremities of another diameter.

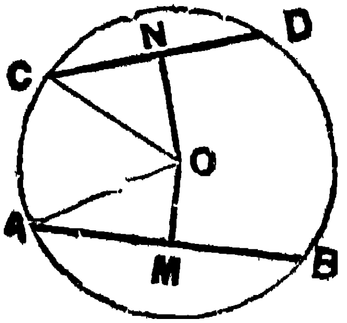
16. Two equal circles intercept equal chords on a straight line; prove that the straight line is either parallel to the line of centres of the circles, or bisects it

*THEOREM 4. [Euc. III. 15.]

In a circle—

(i) The chord which is nearer to the centre is greater than one more remote;

(ii) Conversely, the greater chord is nearer to the centre than the less.



Let AB, CD be chords of a circle whose centre is O; and let OM, ON be perpendiculars on AB and CD from O.

(i) Let AB be nearer to O than CD;
that is, let OM be less than ON.

To prove that AB *is greater than* CD.

Construction. Join OA, OC.

Proof. Because OM is perp. to the chord AB,

$\therefore$ OM bisects AB;

$\therefore AB = 2AM$.

Similarly $CD = 2CN$.

And because OMA and ONC are right angles;

$\therefore OA^2 = OM^2 + AM^2$, and $OC^2 = ON^2 + CN^2$.

Now $OA = OC$; $\therefore OA^2 = OC^2$,

$\therefore OM^2 + AM^2 = ON^2 + CN^2$.

But OM is less than ON; [*Hyp.*

$\therefore OM^2$ is less than ON^2;

$\therefore AM^2$ is greater than CN^2;

$\therefore$ AM is greater than CN;

that is, AB is greater than CD.

(ii) *Conversely*, let AB be greater than CD.

To prove that OM *is less than* ON.

Proof. As before, it may be shewn that

$AB = 2AM$ and $CD = 2CN$

and $OM^2 + AM^2 = ON^2 + CN^2$.

But AB is greater than CD ; [*Hyp.*

∴ AM is greater than CN ;

∴ AM^2 is greater than CN^2 ;

∴ OM^2 is less than ON^2 ;

that is, OM is less than ON.

Q. E. D.

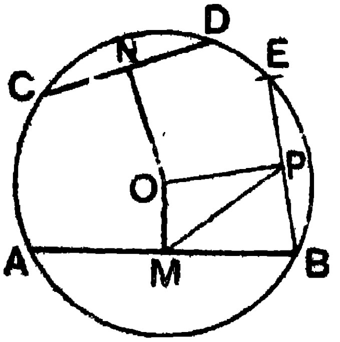
Cor. *The diameter is the* greatest chord *of a circle.*

For the perp. from the centre on a diameter is 0 *i.e.*, *least.*

[Alternative Proof.]

Let AB, CD be chords of a circle whose centre is O ; and let OM, ON be perpendiculars on AB and CD from O.

(i) Let AB be nearer to O than CD ;
that is, let OM be less than ON.



To prove that AB *is greater than* CD.

Construction. With centre B and radius equal to the other chord CD, draw a circle cutting the original circle at E.

Join BE and draw OP perpendicular to BE. Join PM.

Proof. Because OM is perp. to the chord AB,

∴ OM bisects AB ;

∴ $AB = 2BM$.

Similarly $CD = 2CN$ and $BE = 2BP$.

Since the chord $CD =$ the chord BE. [*Const.*

∴ perp. $ON =$ perp. OP. [III. 3.

Now ∵ OM is less than OP,

∴ the $\angle OPM$ is less than the $\angle OMP$. [I. 9.

But the $\angle OPB =$ the $\angle OMB$, being rt. $\angle^s$;

∴ the $\angle BPM$ is greater than the $\angle BMP$,

∴ BM is greater than BP. [I. 10.

∴ AB is greater than BE,

that is, AB is greater than CD.

(ii) *Conversely*, let AB be greater than CD.

To prove that OM *is less than* ON.

Proof. As before, it may be shewn that

$AB = 2AM$, $CD = 2CN$ and $BE = 2BP$; and $ON = OP$.

Because AB is greater than CD and $CD = BE$; [*Hyp.*

∴ BM is greater than BP;

∴ the $\angle BPM$ is greater than the $\angle BMP$. [I. 9.

But the $\angle OPB =$ the $\angle OMB$, being rt. $\angle^s$,

∴ the $\angle OPM$ is less than the $\angle OMP$,

∴ OM is less than OP. [I. 10.

that is, OM is less than ON. Q.E.D.

EXERCISES XLVI.

[*Theoretical.*]

1. In a circle, the greater chord subtends a greater angle at the centre than the less; and *conversely*, the chord which subtends a greater angle at the centre is greater than one which subtends a less angle.

2. In a circle, the chord which is nearer to the centre subtends a greater angle at the centre than one more remote.

3. **The least possible chord which can be drawn through a given point within a circle is that which is perpendicular to the diameter through that point.**

4. P and Q are two points within a circle whose centre is O; if the shortest chords drawn through P and Q are equal, prove that $OP = OQ$.

ANGLE PROPERTIES

DEFINITIONS

1. An **arc** of a circle is any portion of the circumference.

An arc greater than half the circumference is called a **major arc**; and an arc less than half the circumference is called a **minor arc**.

2. Two arcs are said to be **conjugate** to one another when they together make up the whole circumference.

3. A **chord** of a circle is the straight line joining any two points on the circumference.

The chord that joins the extremities of an arc is called the **chord of the arc.**

4. A **segment** of a circle is a portion of the circle which is bounded by an arc and the chord of the arc.

The straight line joining the extremities of the bounding arc of a segment is called the **chord of the segment.**

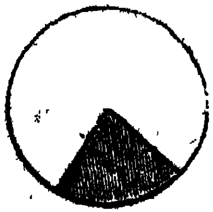
The chord of the segment is also called the **base of the segment.**

If a segment is greater than a semi-circle, it is called a **major segment**; if less, a **minor segment.**

5. An angle **subtended** by an **arc,** or by a **straight line,** at any point is the angle contained by two straight lines drawn from the point to the extremities of the arc, or of the straight line.

6. An **angle in a segment** of a circle is the angle formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord. It is thus an angle which the chord of the segment subtends at any point in the bounding arc.

7. **Similar segments** of circles are those in which the angles are equal.



8. A **sector** of a circle is the figure bounded by two radii and the arc intercepted between them.

9. The **angle of a sector** is the angle between its bounding radii.

THEOREM 5. [Euc. III. 20.]

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

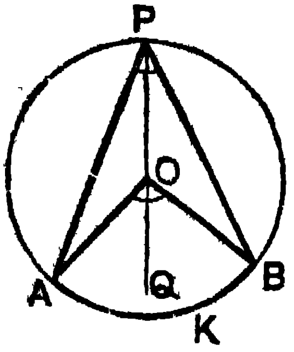


FIG. 1.

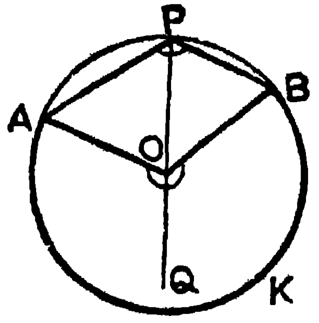


FIG 2.

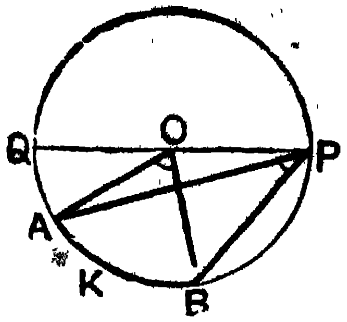


FIG. 3.

Let APB be a circle whose centre is O; and let the arc AKB subtend the $\angle AOB$ at O and the $\angle APB$ at any pt. P on the remaining part of the circumference.

To prove that *the* $\angle AOB = 2\angle APB$.

Construction. Join PO, and produce it to Q.

Proof. Then because $OA = OP$,

$\therefore$ the $\angle OAP =$ the $\angle OPA$,

$\therefore$ the $\angle OAP +$ the $\angle OPA = 2\angle OPA$.

But the ext. $\angle AOQ =$ the $\angle OAP +$ the $\angle OPA$;

[I. 16, *Cor.* 1.

$\therefore$ the $\angle AOQ = 2\angle OPA$.

Similarly the $\angle BOQ = 2\angle OPB$.

$\therefore$ the sum of the $\angle^s$ AOQ, BOQ, [Figs. 1 and 2], or their difference [Fig. 3.], is equal to twice the sum or difference of the $\angle^s$ OPA, OPB;

that is, the $\angle AOB$ (reflex in Fig. 2) $= 2\angle APB$.

Q. E. D.

Note. An angle "subtended by an arc" may be said to "stand on the arc." This Theorem may then be enunciated thus:

The angle at the centre of a circle is double of an angle at the circumference standing on the same arc.

Obs. Suppose the points A and B to move away from each other. Thus the angle AOB increases, and when AO and OB are in one straight line, *i.e.*, when AOB is a diameter, the angle AOB becomes a straight angle and the arc AKB a semi-circumference. Hence a semi-circumference subtends at the centre a straight angle.

Also in Fig. 2, the arc AKB is a major arc and subtends at the centre a reflex angle, *i.e.*, an angle greater than two right angles; and in Fig. 1, it is a minor arc and subtends at the centre an angle less than two right angles.

Hence at the **centre** of a circle,

(i) a major arc subtends a reflex angle;

(ii) a semi-circumference subtends a straight angle;

(iii) a minor arc subtends an angle less than two right angles.

EXERCISES XLVII.

1. Draw an arc of a circle subtending an angle of 75° at the circumference what is the magnitude of the angle that it subtends at the centre? Mark this angle in your figure.

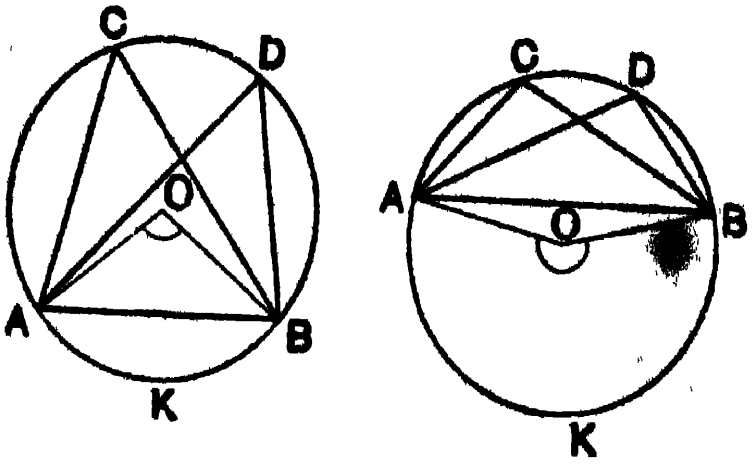
2. Measure the angle that the conjugate arc in Ex. 1 subtends at the circumference. Shew that *the two are together equal to two right angles.*

3. **AB, CD** are two chords of a circle **ACBD** intersecting at **O**, prove that the triangles **AOC, DOB** are equiangular.

4. **AB, CD** are two chords of a circle **ACBD**, intersecting at **E**; prove that the sum of the angles subtended by the arcs **AC, BD** at the centre is twice the $\angle$**AEC**.

THEOREM 6. [Euc. III. 21.]

Angles in the same segment of a circle are equal.



Let ABC be a circle whose centre is O and let ACB, ADB be angles in the same segment ACDB.

To prove that *the* ∠ ACB = *the* ∠ ADB.

Construction. Join OA, OB.

Proof. Because the arc AKB subtends the ∠ AOB at the centre and the ∠ ACB at the circumference ;

∴ the ∠ AOB = 2 ∠ ACB. [III. 5.

Similarly the ∠ AOB = 2 ∠ ADB.

∴ the ∠ ACB = the ∠ ADB.

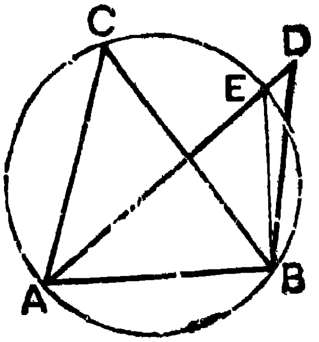
Q. E. D.

Cor. *Angles at the circumference of a circle standing on the same arc are equal.*

THEOREM 7.

[CONVERSE OF THEOREM 6.]

If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.



Let the straight line AB, joining two pts. A and B, subtend equal angles ACB, ADB at two other pts. C and D, on the same side of AB.

To prove that A, B, D, C *lie on a circle.*

Proof. Let a circle be drawn through A, B and C.

If this circle does not pass through D, let it cut AD or AD produced, at the pt. E.

Join EB.

Then the ∠ ACB = the ∠ AEB in the same segment.

But the ∠ ACB = the ∠ ADB ; [*Hyp.*

∴ the ∠ AEB = the ∠ ADB ;

that is, an exterior angle is equal to an interior opposite angle, which is absurd.

∴ the circle through A, B and C must pass through D ;

thus A, B, D, C lie on a circle.

Q. E. D.

Cor. *The locus of a point at which a given straight line subtends a given angle is an arc of a circle of which the given line is the chord.*

Def. Points which lie on the circumference of the same circle are said to be **concyclic.**

EXERCISES XLVIII.

[*Theoretical.*]

1. **AB** is a fixed chord of a circle and **P** any point on the arc of a segment cut off by **AB**; prove that the $\angle$**ABP** + the $\angle$**BAP** is constant.

2. Two tangents **AB, AC** are drawn to a circle from a fixed point **A**; **P** is any point on the arc outside the $\triangle$**ABC**; prove that

the $\angle$**ABP** + the $\angle$**ACP** is constant.

3. Two circles cut at **A** and **B**; prove that any straight line drawn through **A** and terminated by the circles subtends a constant angle at **B**.

4. Omitted.

5. Two circles intersect at **P** and **Q**; any two straight lines **APB, CPD** are drawn through **P** and terminated by the circles; shew that the arcs **AC, BD** subtend equal angles at **Q**.

6. **APB** is a fixed straight line passing through **P**, a point of intersection of two circles and terminated by them; **QPR** is any other line passing through **P** and terminated by the circles; prove that **AQ, RB** when produced intersect at a constant angle.

7 **Similar segments of circles on equal chords are equal to one another** [Euc. III. 24.]

[Apply one segment to the other and then prove by *reductio ad absurdum* that they are equal.]

8. The locus of the vertex of a triangle on a given base and having a given vertical angle is an arc of a circle.

9. Find the locus of the vertex of a triangle, having given the base and the sum of the angles at the base.

10. **AB** is a fixed chord of a circle and **APB** is a triangle inscribed in it; perpendiculars are drawn from **A** and **B** to **PB** and **PA** respectively intersecting at **O**; find the locus of **O**.

THEOREM 8. [Euc. III. 31.]

The angle in a semi-circle is a right angle; the angle in a segment greater than a semi-circle is less than a right angle; and the angle in a segment less than a semi-circle is greater than a right angle.

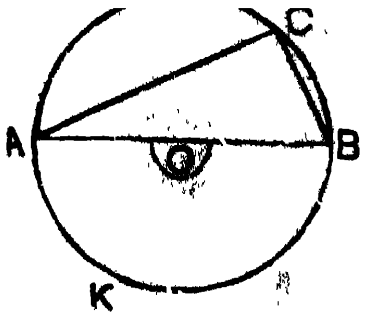


Fig. 1.

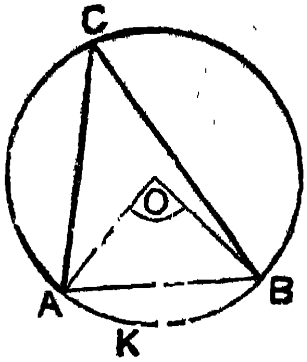


Fig. 2.

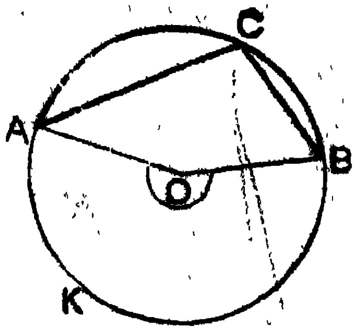


Fig. 3.

Let AKBC be a circle of which O is the centre.

(i) Let ACB [Fig. 1] be a semi-circle.

To prove that *the* $\angle$ ACB *is a right angle.*

Proof. Because ACB is a semi-circle;

∴ the arc AKB is a semi-circumference;

∴ the $\angle$AOB at the centre subtended by the arc AKB is a *straight angle, i.e., 2 rt. angles.*

But the $\angle$ACB at the circumference is half the $\angle$AOB at the centre, subtended by the same arc AKB; [III. 5.

∴ the $\angle$ACB = one rt. angle.

(ii) Let ACB [Fig. 2] be a segment greater than a semi-circle.

To prove that *the* $\angle$ ACB *is less than a right angle.*

Construction. Join OA, OB.

Proof. Because ACB is a major segment;

∴ the arc AKB is a minor arc;

∴ the $\angle$AOB at the centre subtended by the arc AKB is less than 2 rt. angles.

But the $\angle ACB$ at the circumference is half the $\angle AOB$ at the centre, subtended by the same arc AKB ;

$\therefore$ the $\angle ACB$ is less than one rt angle.

(iii) Let ACB [Fig. 3] be a segment less than a semi-circle.

To prove that *the* $\angle ACB$ *is greater than a right angle.*

Construction. Join OA, OB.

Proof. Because ACB is a minor segment ;

$\therefore$ the arc AKB is a major arc ;

$\therefore$ the $\angle AOB$ at the centre subtended by the arc AKB is greater than 2 rt. angles.

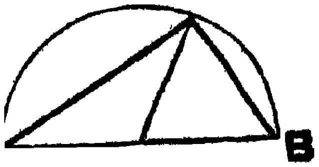
But the $\angle ACB$ at the circumference is half the $\angle AOB$ at the centre, subtended by the same arc AKB ;

$\therefore$ the $\angle ACB$ is greater than one rt. angle.

Q E.D.

An *alternative proof* of the first part of Theorem 8.

The angle in a semi-circle is a right angle.



Let ACB be a semi-circle of which the diameter is AB and centre O.

Let C be any point on the arc ACB.

To prove that *the* $\angle ACB$ *is a right angle.*

Construction. Join OC.

Proof. Because $OA = OC$, $\therefore$ the $\angle OCA =$ the $\angle OAC$;

and because $OB = OC$, $\therefore$ the $\angle OCB =$ the $\angle OBC$.

$\therefore$ the whole $\angle ACB =$ the $\angle OAC +$ the $\angle OBC$;

$\therefore$ 2 $\angle ACB =$ the $\angle ACB +$ the $\angle BAC +$ the $\angle ABC$

$= 2$ rt. angles.

$\therefore$ the $\angle ACB =$ one rt. angle. Q.E.D.

EXERCISES XLIX.

[*Graphical and Theoretical.*]

1. What is the magnitude of an arc of a circle which subtends a right angle (i) at the centre and (ii) at the circumference?

2. Draw an arc of a circle subtending an angle of 60° at the circumference. Shew that it is one-third of the circumference.

3. **A circle described on the hypotenuse of a right-angled triangle as diameter passes through the vertex of the right angle.**

4. The middle point of the hypotenuse of a right-angled triangle is equidistant from the three angular points.

5. The sides of a triangle are as $1 : 2 : \sqrt{3}$; prove that the circle described on the greatest side as diameter passes through the opposite angular point.

6. A variable straight line passes through a fixed point. Find the locus of the foot of the perpendicular drawn to it from another fixed point.

7. **ABC** is a triangle; perpendiculars **AM**, **BN** are drawn from **A** and **B** to the opposite sides; prove that the middle point of **AB** is equidistant from **M** and **N**.

8. Circles described on any two sides of a triangle as diameters intersect on the third side, or that side, produced.

9. If two circles cut one another at **A** and **B** and from **A**, diameters **AP**, **AQ** are drawn, one in each circle, prove that **P**, **B**, **Q** are collinear.

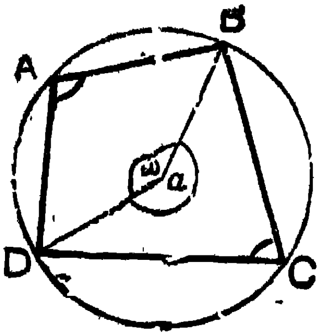
10. Omitted.

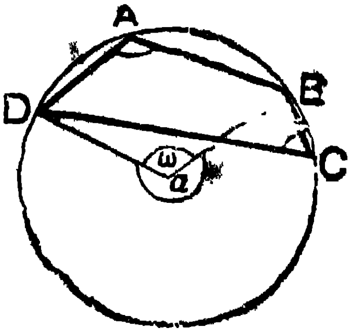
11. **Chords of a circle whose centre is O pass through a fixed point P, either within, on or without the circle; shew that the locus of their middle points is a circle described on OP as diameter.**

12. **AB** is a diameter of a circle; **C** and **D** are two points on the circumference such that the arc **CD** is constant; **AD**, **BC** are joined intersecting at **P**; shew that the ∠**APB** is constant.

THEOREM 9. [Euc. III. 22.]

The opposite angles of any quadrilateral inscribed in a circle are supplementary.





Let ABCD be a quadrilateral inscribed in the circle ABC.

To prove that

(i) *the* $\angle BAD$ + *the* $\angle BCD = 2$ *right angles* ;

(ii) *the* $\angle ABC$ + *the* $\angle ADC = 2$ *right angles.*

Construction. Join B, D to the centre of the circle.

Proof. Let α (alpha) and ω (omega) be the angles which the arcs BCD and BAD subtend at the centre.

Because the $\angle BAD$ at the circumference = half the $\angle \alpha$ at the centre subtended by the same arc BCD ;

and the $\angle BCD$ at the circumference = half the $\angle \omega$ at the centre subtended by the same arc BAD ;

$\therefore$ the $\angle BAD$ + the $\angle BCD = \frac{1}{2}$ of the $\angle$s α and ω

$= \frac{1}{2}$ of 4 rt. angles = 2 rt. angles.

Similarly the $\angle ABC$ + the $\angle ADC = 2$ rt. angles.

Q. E. D.

Cor. *If a side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite interior angle.*

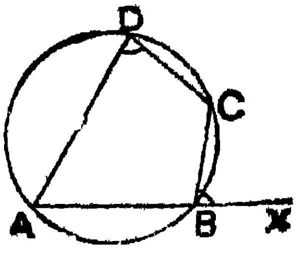
Let **ABCD** be a cyclic quadl. and let a side **AB** be produced to **X**.

To prove that the $\angle CBX$ = the opp. int. $\angle D$.

Proof. The $\angle CBX$ + the $\angle CBA = 2$ rt. $\angle$s.

Also the $\angle D$ + the $\angle CBA = 2$ rt. $\angle$s.

$\therefore$ the $\angle CBX$ = the $\angle D$.

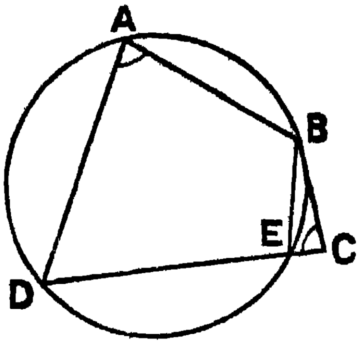


Def. If all the vertices of a rectilineal figure lie on the circumference of the same circle, the rectilineal figure is said to be **cyclic**.

THEOREM 10.

[CONVERSE OF THEOREM 9.]

If a pair of opposite angles of any quadrilateral are supplementary, the quadrilateral is cyclic.



Let ABCD be a quadrilateral, in which the opposite angles A and C are supplementary.

To prove that ABCD *is a cyclic quadrilateral.*

Proof. Let a circle be drawn through A, B and D.

If it does not pass through C, let it cut CD, or CD produced, at the pt. E. Join EB.

Because ABED is a cyclic quadl. ;

∴ the ∠A is the supplement of the ∠BED.

But the ∠A is the supplement of the ∠C. [*Hyp.*

∴ the ∠BED = the ∠C ;

that is, an exterior angle is aqual to an interior opposite angle, which is absurd ;

∴ the circle through A, B, D must pass through C ;

that is, ABCD is a cyclic quadrilateral.

Q.E.D.

Cor. *If the exterior angle formed by producing a side of a quadrilateral is equal to the opposite interior angle, the quadrilateral is cyclic.*

Let **AB** be produced to **X**; and let the ∠**CBX** = the ∠**ADC** [fig. III. 9, *Cor.*]

Then ∵ the ∠s **CBX, ABC** are supplementary; ∴ the ∠s **ADC, ABC** are supplementary; ∴ the quadl. **ABCD** is cyclic.

EXERCISES L.

[*Theoretical*]

1. If a parallelogram is cyclic, it must be a rectangle.

2. A triangle is inscribed in a circle; shew that the angles in the three segments exterior to the triangle are together equal to four right angles

3. Two circles intersect at **A** and **B** and any two lines **PAQ, RBS** are drawn through **A, B** to cut the circles; prove that **PR** is parallel to **QS**.

4. **If the bisectors of any two opposite angles of a cyclic quadrilateral cut the circle at A and B, prove that AB is a diameter of the circle.**

5. The internal bisector of any angle of a cyclic quadrilateral and the external bisector of the opposite angle meet on the circle.

6. Omitted.

7. Two circles intersect at **A** and **B**, and through **B** any straight line **PBQ** is drawn to cut the circles in **P** and **Q**. Prove that **AP** and **AQ** subtend equal angles at the centres.

8. The opposite sides of a cyclic quadrilateral are produced to meet at **A** and **B**; if about the triangles thus formed, outside the quadrilateral circles are described intersecting at **C**, shew that **A, B, C**, are collinear.

9. **If in two circles, equal chords subtend equal or supplementary angles at the circumfence, the circles are equal.** [Use (XLVIII), 7.]

10. A line, which makes equal angles with one pair of opposite sides of a cyclic quadrilateral, makes equal angles with the other pair and with the diagonals.

11. **The internal or external bisectors of the angles of any convex quadrilateral form a cyclic quadrilateral.**

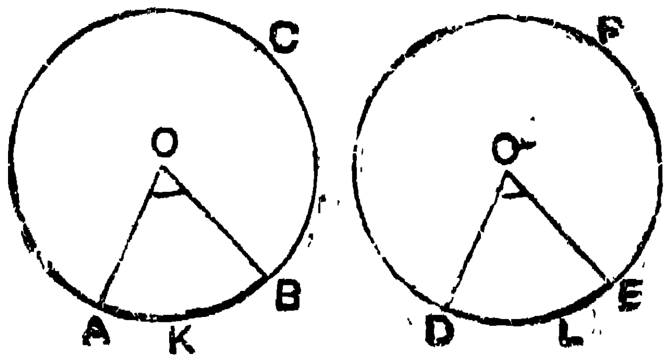
12. **The three altitudes of a triangle are concurrent.**

THEOREM 11. [Euc. III. 26 and 27.]

In equal circles (or in the same circle),—

(i) If two arcs subtend equal angles at the centres, they are equal;

(ii) Conversely, if two arcs are equal, they subtend equal angles at the centres.



Let ACB, DFE be two equal circles, whose centres are O and O′.

(i) Let the ∠AOB = the ∠DO′E.

To prove that *the arc* AKB = *the arc* DLE.

Proof. Apply the circle ACB to the circle DFE, so that O falls on O′ and OA along O′D.

Then because the ∠O = the ∠O′,

∴ OB falls along O′E.

And the circles being equal, their radii are equal;

∴ A coincides with D, and B with E, and the circumferences of the two circles wholly coincide;

∴ the arc AKB coincides with the arc DLE;

that is, the arc AKB = the arc DLE.

(ii) *Conversely*, let the arc AKB = the arc DLE.

To prove that *the* ∠AOB = *the* ∠DO′E.

Proof. Apply the circle ACB to the circle DFE, so that O falls on O′ and OA along O′D.

Now the circles being equal, their radii are equal;

∴ A coincides with D and the two circumferences wholly coincide.

And because the arc AKB = the arc DLE;

∴ B coincides with E, and consequently OB with O′E.

∴ the ∠AOB = the ∠DO′E.

Q.E.D.

Cor. *In equal circles, the sectors on equal arcs are equal.*

EXERCISES LI.

[*Theoretical*]

1. **In equal circles (or in the same circle), if one arc subtends at the centres an angle** *double* **of that which another arc subtends, then the former arc is** *double* **of the latter.**

2. State and prove the Converse of the theorem in Ex. 1.

3. AB is a diameter of a circle and PQ a chord parallel to it; prove that the arc AP = the arc BQ.

[Bisect AB at O; and join OP, OQ. Prove ∠AOP = ∠BOQ.]

4. AB is a diameter and AC a chord of a circle ABC whose centre is O; OP is drawn parallel to AC, meeting the arc AB in P; prove that the arc BP = the arc CP.

5. P is a point on the circumference of a circle whose centre is O; if P is equidistant from the radii OA, OB, prove that it is the middle point of the arc AB.

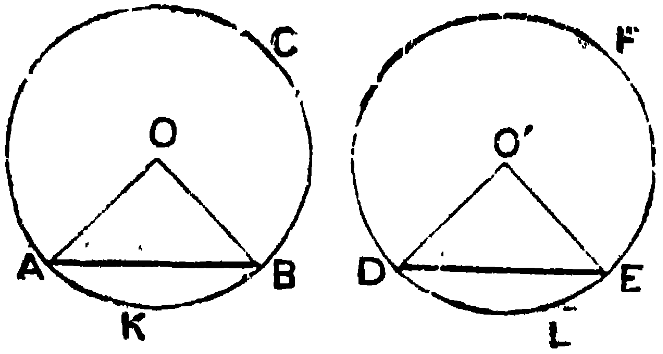
6. ACBD is a semi-circle of which AB is the diameter and CD a chord; AB and CD are produced to meet at E; if AB = 2DE, prove that arc AC = 3 arc BD.

THEOREM 12. [Euc. III. 28 and 29.]

In equal circles (or in the same circle),—

(i) If two chords are equal, they cut off equal arcs, the major arc equal to the major arc and the minor to the minor;

(ii) Conversely, if two arcs are equal, the chords of the arcs are equal.



Let ABC, DEF be two equal circles, whose centres are O,O′.

(i) Let the chord AB = the chord DE.

To prove that *the major arc* ACB = *the major arc* DFE,

and *the minor arc* AKB = *the minor arc* DLE.

Construction. Join OA, OB ; O′D, O′E.

Proof. In the $\triangle^s$ OAB, O′DE,

(i) OA = O′D
(ii) OB = O′E } (radii of equal circles),
and (iii) AB = DE ; [*Hyp.*

∴ the ∠O = the ∠O′; [I. 7.

∴ the arc AKB = the arc DLE ; [III. 11.

and these are the minor arcs.

Also the whole circumference ABC

= the whole circumference DEF ;

∴ the remaining arc ACB = the remaining arc DFE ;

and these are the major arcs.

(ii) *Conversely*, let the arc AKB = the arc DLE.

To prove that *the chord* AB = *the chord* DE.

Construction. Join OA, OB ; O'D, O'E.

Proof. Because the arc AKB = the arc DLE,

∴ the $\angle$ O = the $\angle$ O'. [III. 11.

Then in the $\triangle^s$ OAB, O'DE,

$$\left.\begin{array}{l}\text{(i) } OA = O'D \\ \text{(ii) } OB = O'E\end{array}\right\} \text{(radii of equal circles),}$$

and (iii) the $\angle$O = the $\angle$O' ; [*Proved.*

∴ the chord AB = the chord DE. [I. 4.

Q. E. D.

Cor. *In equal circles, sectors having equal angles are equal.*

EXERCISES LII.

[*Theoretical*]

1. **In equal circles (or in the same circle), equal chords subtend equal angles at the centres** ; and conversely, **chords that subtend equal angles at the centres are equal.**

2. The diagonals of a quadrilateral inscribed in a circle are equal, if two of its opposite sides are equal.

3. A pair of opposite sides of a quadrilateral inscribed in a circle are equal, if its diagonals are equal.

4. ABC is an isosceles triangle inscribed in a circle ; if DE, the perpendicular bisector of the base BC, meets the arc BC in E, prove that the arc ABE is half the circumference.

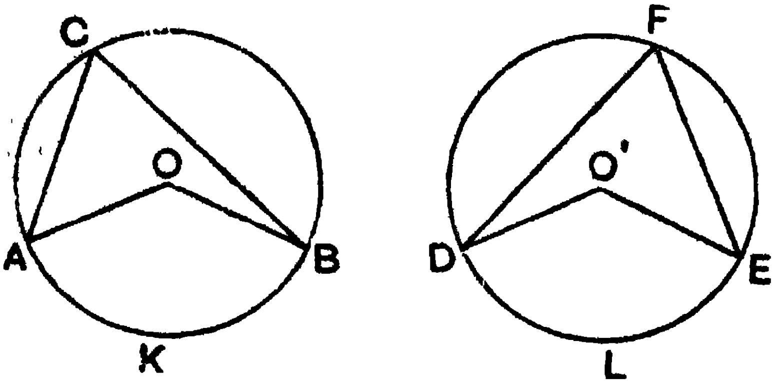
*THEOREM 12A.

[EXTENSION OF THEOREM 11.]

In equal circles (or in the same circle),—

(i) if two arcs subtend equal angles at the centres or at the circumferences, they are equal.

(ii) Conversely, if two arcs are equal, they subtend equal angles at the centres or at the circumferences.



Let ACB, DFE be two equal circles, whose centres are O, O'.

(i) Let the ∠AOB = the ∠DO'E at the centres,
or the ∠ACB = the ∠DFE at the circumferences.

To prove that *the arc* AKB = *the arc* DLE.

Proof. Apply the ⊙ ABC to the ⊙DFE, so that O falls on O' and OA along O'D.

Then because the ∠O = the ∠O',
∴ OB falls along O'E.

And the circles being equal, their radii are equal;

∴ A coincides with D and B with E, and the circumferences of the circles wholly coincide;

∴ the arc AKB coincides with the arc DLE,
that is, the arc AKB = the arc DLE.

Again the ∠AOB at the centre = 2 ∠ACB at the circumference;

and likewise the ∠DO′E = 2 ∠DFE.

But the ∠ACB = the ∠DFE;

∴ the ∠AOB = the ∠DO′E;

∴ the arc AKB = the arc DLE.

(ii) *Conversely*, let the arc AKB = the arc DLE.

To prove that *the* ∠AOB = *the* ∠DO′E;

also the ∠ACB = *the* ∠DFE.

Proof. Apply the ⊙ACB to the ⊙DFE, so that O falls on O′ and OA along O′D;

Now the circles being equal, their radii are equal;

∴ A coincides with D and the two circumferences wholly coincide.

And because the arc AKB = the arc DLE; [*Hyp.*

∴ B coincides with E and consequently OB with O′E;

∴ the ∠AOB = the ∠DO′E.

And the ∠ACB at the circumference = $\frac{1}{2}$ ∠AOB,

and likewise the ∠DFE = $\frac{1}{2}$ ∠DO′E,

∴ the ∠ACB = the ∠DFE.

Q.E.D.

This Theorem may be otherwise enunciated thus:

In equal circles (or in the same circle), equal angles at the centres or at the circumferences stand on equal arcs; and conversely, the angles at the centres or at the circumferences which stand on equal arcs are equal.

EXERCISES LIII.

1. **In equal circles (or in the same circle), equal chords subtend equal angles at the centres or at the circumferences;** and conversely, **chords that subtend equal angles at the centres or at the circumferences are equal.**

2. Two equal circles intersect at **A** and **B**, and through **A**, any straight line is drawn to cut them in **P** and **Q**; prove that **BP = BQ**

3. The straight lines joining the extremities of two parallel chords of a circle are equal.

[Observe that there are two cases.]

4. The straight lines joining the ends of two equal chords of a circle are either equal or parallel.

5. **Parallel chords of a circle intercept equal arcs,** and conversely, **chords of a circle which intercept equal arcs are parallel.**

6. The oblique sides and the diagonals of a trapezium inscribed in a circle are equal.

7. A quadrilateral inscribed in a circle is a trapezium, (i) if a pair of opposite sides are equal or (ii) if its diagonals are equal.

8. **If two chords of a circle intersect *within* or *without* it, the angle between them is equal to the angle at the circumference standing on an arc equal to the *sum* or *difference* of the intercepted arcs.**

9. Two circles intersect at **A** and **B**, and **P** is any point on one of them; if **PA, PB** are produced to cut the other circle in **K** and **L**, prove that the chord **KL** is constant.

10. **AB is a fixed chord of a circle and P any point on one of the arcs cut off by it; prove that the bisector of the angle APB passes through a fixed point.**

11. If two chords of a circle intersect at right angles, the sum of the opposite arcs they intercept is equal to the semi-circumference.

[This follows at once from Ex. 8.]

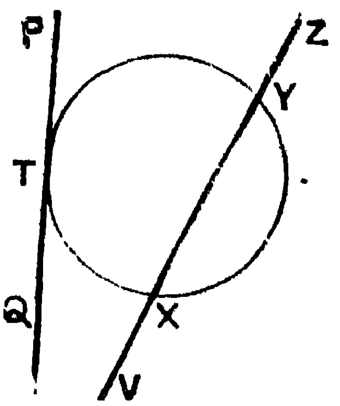
12. Parallel chords of a circle at the extremities of a diameter meet the circumference at the extremities of another diameter.

13. **The internal and external bisectors of the vertical angle of a triangle inscribed in a circle meet the circle at the extremities of the diameter which is perpendicular to the base; and the converse.**

TANGENT PROPERTIES

1. *Def.* A **tangent** to a circle is the straight line which meets the circle at one point only and being produced does not cut it. Thus **PQ** is a tangent.

Hence a tangent has *one point, and one only*, in common with the circle; and consequently, a tangent lies wholly without the circle.

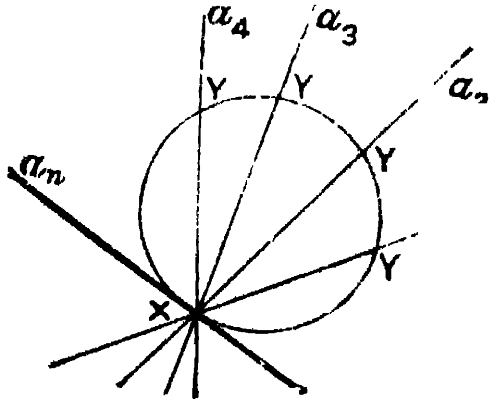


A tangent is said to **touch** the circle, and the point where it meets the circle is called the **point of contact**. Thus **T** is the point of contact.

2. *Def.* A **secant** of a circle is a straight line of indefinite length which cuts the circle at two points. Thus **VXYZ** is a secant.

It is evident that the part of a secant intercepted between the circumferences is a **chord.**

3. *Tangent is the "limiting" position of secant.* Suppose the secant **XY** to be turned about **X**. Thus while **X** remains fixed, **Y** moves along the curve towards **X**, and the secant **XY** assumes different positions, such as a_1, a_2, a_3, ...Hence ultimately when **Y** moves up to **X** and coincides with it, the secant **XY** takes the position a_n, and becomes a tangent to the curve at **X**. Thus the tangent at **X** may be regarded as the "limiting" position of the secant through **X**, when the other point of intersection moves into coincidence with **X**.



Two points on a curve are said to be *consecutive*, when there is absolutely no interval between them. Thus when **Y** coincides with **X**, **X** and **Y** are said to be two consecutive points on the circle. Hence we have the alternative

Def. **A tangent to a curve is a straight line which meets the curve at two consecutive points.**

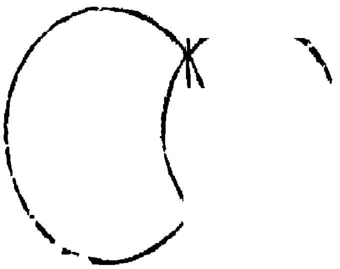
4. *Def.* Two circles which meet at one point only, are said to **touch** one another, and the common point where they meet is called the **point of contact.**

Since two touching circles have one point in common, it is evident that they cannot cut one another, for then they would have *two* points in common.

When one circle is wholly within the other, they are said to touch **internally** or to have **internal contact**, as in Fig. 1, Theorem 16.

When one circle is wholly without the other, they are said to touch **externally** or to have **external contact,** as in Fig. 2, Theorem 16.

5. Suppose two circles to cut one another at P and Q. Let P remain fixed and one of the circles turn about P, so that Q continually approches P. Then when ultimately Q coincides with P the two circles touch one another at P, [see figures, Theorem 16] and at the same time PQ becomes the tangent to both circles at P. Thus



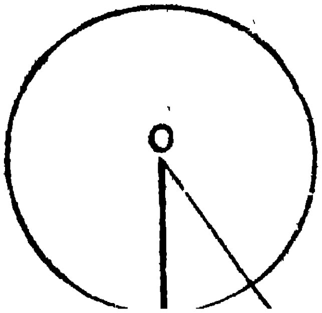
Two circles which touch one another, whether internally or externally, have a common tangent at their point of contact.

And conversely

If two circles have a common tangent at a point where they meet, they touch one another at that point. (Prove this.)

THEOREM 13. [Euc. III. 18.]

The tangent at any point of a circle is perpendicular to the tangent drawn through the point.



Let AT be a tangent at the point A to a circle of which the centre is O ; and let OA be the radius drawn through A.

To prove that AT *is perpendicular to the radius* OA.

Construction. Take any point P in AT. Join OP.

Proof. Because a tangent meets the circle at only one point, every point in AT except A is outside the circle ;

∴ OP is greater than the radius OA,

that is, OA is less than *any* line that can be drawn from O to AT.

∴ OA is the shortest line from O to AT ;

∴ OA is perpendicular to AT. [I. 12, *Cor.*

Q. E. D.

Cor. 1. *The perpendicular to a radius at the point where the radius meets the circle is a tangent to the circle.*

[Euc. III. 16.]

For the tangent at **A** is perp. to **OA** ; therefore any other line perp. to **OA** at **A** must coincide in position with the tangent.

Cor. 2 *One, and only one, tangent can be drawn to a circle at a given point on its circumference.*

For there can be *only one* perpendicular to **OA** at the pt. **A**.

THEOREM 13.

[ALTERNATIVE PROOF BY THE METHOD OF LIMITS.]

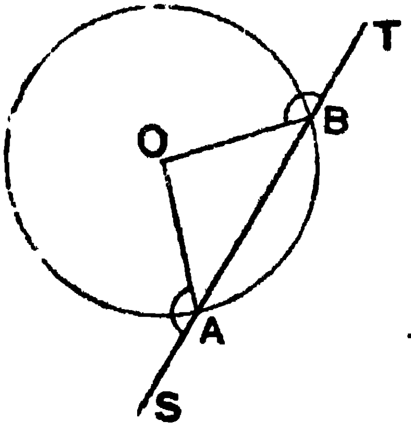


FIG. 1.

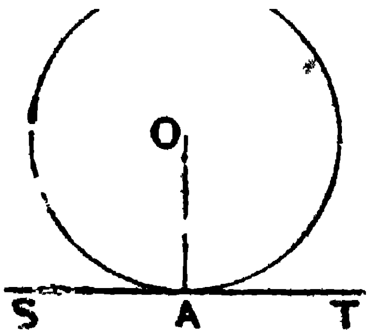


FIG. 2.

Let A be a point on a circle of which the centre is O.

To prove that *the tangent at* A *is perpendicular to the radius* OA.

Construction. Draw a secant SABT [Fig. 1] cutting the circle at A and B.

Join OA, OB.

Proof. Because $OA = OB$;

$\therefore$ the $\angle OAB =$ the $\angle OBA$;

$\therefore$ their supplements are equal;

$\therefore$ the $\angle OAS =$ the $\angle OBT$;

and this is always true for any position of B.

Now suppose the secant ST to be turned about A, so that B moves along the arc towards A.

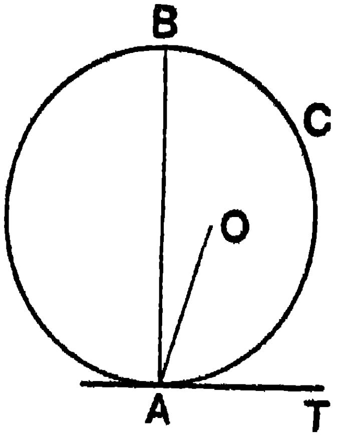
Thus when B moves up to A, and ultimately coincides with it [Fig. 2], then (i) the secant ST becomes the tangent at A, and (ii) OB coincides with OA.

But when OB coincides with OA, the equal angles OAS, OBT become adjacent angles, and therefore right angles;

$\therefore$ OA is perpendicular to ST. Q.E.D.

THEOREM 14. [Euc. III. 19.]

The perpendicular to the tangent at the point of contact passes through the centre.



Let AT be the tangent at the point A of a circle ABC, and let AB be the perpendicular to AT at A.

To prove that AT *passes through the centre of the circle* ABC.

Proof. For if AB does not pass through the centre, let O be the centre of the circle.

Join OA.

Because AT is a tangent to the circle and OA is drawn from the centre O to the point of contact A,

∴ the ∠TAO is a rt. ∠. [III. 13.

But the ∠TAB is a rt. ∠ [*Hyp.*

∴ the ∠TAO = the ∠TAB;

the part equal to the whole, which is absurd,

∴ the centre cannot be otherwise than in AB,

that is, AB passes through the centre.

Q.E.D.

EXERCISES LIV.

[*Theoretical and Graphical.*]

1. **The tangents to a circle at the extremities of a diameter are parallel**; and conversely, **if two tangents to a circle are parallel, the join of their points of contact is a diameter.**

2. With a ruler, draw a tangent **AP** to a circle of radius 6 cm., and a radius to **P**, the point of contact; find the length of **AP**, if the distance of **A** from the centre **O** is 10 cm. Verify by measurement.

3. In Ex. 2, find (i) **OA**, if **AP**$=4$ cm. and **OP**$=3$ cm.; and (ii) **OP**, if **OA**$=13$ cm. and **AP**$=12$ cm. Check your calculation, by measurement.

4. Draw two concentric circles of radii ·5″ and 1·3″ respectively; place in the larger circle three chords each 2·4″ long; shew graphically that each is a tangent to the inner circle.

5. Draw a number of chords, each 6 cm. in length. of a circle whose radius$=5$ cm.; shew that they all touch a concentric circle of radius$=4$cm.

6. **In two concentric circles, all chords of the outer circle which touch the inner are equal and bisected at the points of contact.**

7. **Find the locus of the middle points of equal chords of a circle.**

8. **Find the locus of the centres of circles which touch a given straight line at a given point.**

9. **Find the locus of the centres of circles of given radius which touch a given straight line.**

10. **AT is a tangent to a circle at A; find the locus of the middle points of all chords parallel to AT.**

11. Find the locus of a point from which the tangents drawn to a given circle are of given length.

12. Find a point in a given straight line such that the tangents from it to a given circle are of constant length. When is this problem impossible?

13. **PQ** is a chord and **PT** a tangent to a circle whose centre is **O**; prove that the $\angle$**POQ**$=2\ \angle$**TPQ**

14. In the greater of two concentric circles, chords **PA**, **PB** are drawn to touch the inner circle at **T**, **V**; prove that **TV**$=\frac{1}{2}$ **AB**.

15. **AB** is a diameter and **AC** a chord of a circle; **AP** bisect the $\angle$**BAC**, meeting the circle at **P**; **PQ** is drawn perpendicular to **AC** produced; prove that **PQ** is a tangent to the circle at **P**.

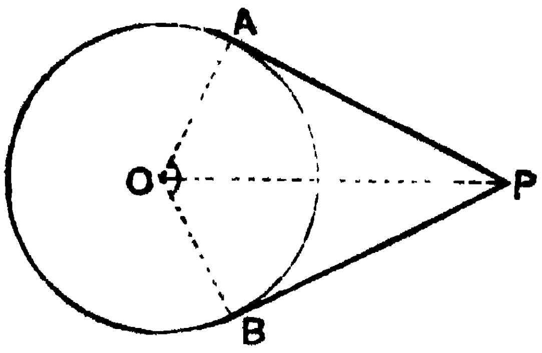
THEOREM 15.

If two tangents are drawn to a circle from an external point,

(i) they are equal ;

(ii) they subtend equal angles at the centre of the circle ;

(iii) they make equal angles with the straight line joining the given point to the centre.



Let PA, PB be two tangents drawn from an external point P to a circle whose centre is O.

To prove that

(i) PA = PB ;
(ii) *the* $\angle$ AOP = *the* $\angle$ BOP ;
and (iii) *the* $\angle$ APO = *the* $\angle$ BPO.

Proof. Since PA is the tangent at A and OA is the radius drawn to A ;

$\therefore$ the $\angle$ PAO is a right angle. [III. 13.

Similarly the $\angle$ PBO is a right angle.

Then in the right-angled $\triangle^s$ PAO, PBO,

(i) the hypotenuse PO is common,
and (ii) one side OA = one side OB, being radii ;

$\therefore$ the triangles are congruent ; [I. 20.

$\therefore$ (i) PA = PB,
(ii) the $\angle$ AOP = the $\angle$ BOP,
and (iii) the $\angle$ APO = the $\angle$ BPO. Q.E.D.

Cor. **PO** *bisects the chord of contact* **AB** *at right angles.*

Def. The straight line joining the points of contact of a pair of tangents is called the **chord of contact.**

EXERCISES LV.

[*Graphical and Numerical.*]

PA is a tangent from a point **P** outside a circle whose centre is **O**; **OP** cuts the circle at **C** :—

1. If **CP** = 8 cm. and **PA** = 12 cm., find the radius. Verify by measurement.

2. If the distance of the chord of contact from **O** = 9″ and **OP** = 2·5″, find (i) **PA**, (ii) the chord of contact and (iii) the radius of the circle. Check the calculations.

3. If the distance of the chord of contact from **O** = 8 cm. and **OP** = 18 cm., find (i) **PA** and (ii) the radius of the circle.

4. If the chord of contact = 24 cm and its distance from **P** = 16 cm., find **AC**.

[*Theoretical.*]

5. **PA, PB** are two tangents to a circle of centre **O**; prove that the angles **APB** and **AOB** are supplementary.

6. **The tangents to a circle at the extremities of a chord make equal angles with the chord.**

7. **The tangents to a circle at the ends of a chord include an angle double of that between the chord and the diameter through either point of contact.**

8. Two circles touch externally at **A**; shew that their common tangent is bisected by the tangent at **A**.

9. From the centre **O** of a circle, **OM** is drawn perpendicular to a chord **AB**; prove that the tangents at **A** and **B** meet on **OM** produced.

10. If a quadrilateral is circumscribed about a circle, prove that the sum of one pair of opposite sides is equal to the sum of the other pair.

11. If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, prove that the quadrilateral can be circumscribed about a circle. [Converse of Ex. 10.]

12. If a polygon of an even number of sides (*e. g.* hexagon) is circumscribed about a circle, shew that the sum of one set of alternate sides is equal to the sum of the remaining set of alternate sides.

13. Prove that the rhombus is the only parallelogram that can be circumscribed about a circle.

14. A quadrilateral is circumscribed about a circle; shew that the angles subtended at the centre by any pair of opposite sides are supplementary.

15. **TA, TB** are two fixed tangents to a circle of centre **O**; **PQ** is any other tangent cutting **TA, TB** in **P** and **Q**; prove (i) that the perimeter of the Δ**TPQ** is constant, and (ii) that **PQ** subtends a constant angle at the centre of the circle.

[Observe that **PQ** subtends a constant angle at the centre of the circle, whether the third tangent cuts **TA, TB** on these tangents *produced*, at **P** and **Q**.]

16. **Find the locus of the centres of circes which touch two intersecting straight lines.**

17. Draw a circle of a given radius to touch two intersecting straight lines.

THEOREM 16. [Euc. III. 11 and 12.]

If two circles touch one another, the point of contact lies on the straight line through the centres.

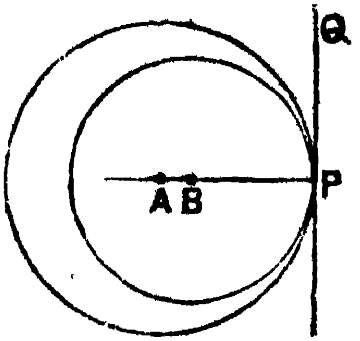


Fig. 1.

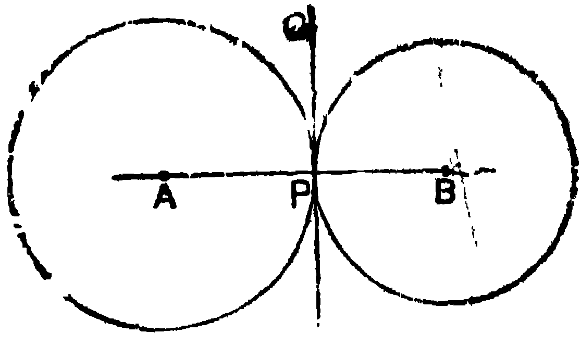


Fig. 2.

Let two circles whose centres are A and B touch at the point P.

To prove that P *lies on the straight line* AB.

Construction. Join AP, BP.

Proof. Because the circles touch at P, they have a common tangent at P.

Let PQ be the common tangent.

Then because AP and BP are radii drawn to the pt. of contact;

∴ the angles APQ and BPQ are rt. angles. [III. 13.

Hence AP BP are both perp. to PQ at the pt. P;

∴ AP, BP are in the same st. line;

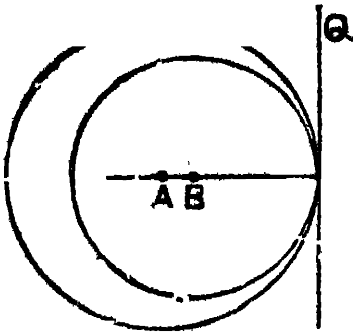
that is, P lies on the st. line AB. Q. E. D.

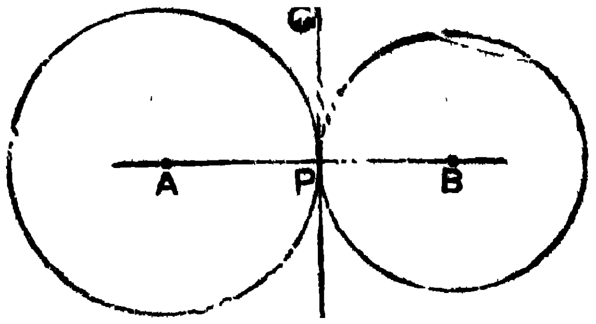
Cor. *If two circles touch one another, the distance between their centres is equal to the* **sum** *or the* **difference** *of their radii, according as they touch* **externally** *or* **internally**.

*THEOREM 16 A.

[Converse of Theorem 16.]

If two circles meet at one point on the line of centres, they touch one another at that point.





Let two circles whose centres are A and B meet at the point P on the line AB.

To prove that *they touch one another at* P.

Construction. At P, draw PQ perp. to AB.

Proof. Then because PQ is perp. to the radius AP,

∴ PQ touches at P the circle of centre A.

For the same reason PQ touches at P the circle of centre B.

Thus PQ is a common tangent to the two circles at P;
∴ the circles touch one another at P. Q. E. D.

EXERCISES LVI.

1. Draw diagrams to shew all the possible relative positions of two circles which have, or have no, contact with one another.

2. Draw two circles of radii 6 and 4 cm. having (i) internal contact and (ii) external contact, and find the distance between their centres in each case.

3. Draw three circles, touching one another externally whose radii are 3, 4 and 5 cm., and find the distances between their centres.

4. Draw three circles, two of radii 2·7 cm. and 1·8 cm. touching each other externally, and both of them touching the third circle of radius 3·4 cm. internally; find the distances between their centres.

5. State, without actually drawing the figures, whether the circles touch, cut or do not cut at all, whose (i) radii are 2·3″, 1·7″, and distance between centres is 4″; (ii) radii 2·3″, 1·7″ distance between centres ·6″; (iii) radii 2·9 cm., 1·6 cm., distance between centres ·5 cm.; (iv) radii

3·8 cm, 2·5 cm., distance between centres 4·3 cm.; (v) radii 5 cm., 3·4 cm., distance between centres 9 cm.

Draw diagrams of (iii) and (v).

6. Draw two circles of radii 3·5 and 4·1 cm. such that the distance between their centres is 0. Can they touch?

7. If two circles cut one another, the distance between their centres is less than the sum but greater than the difference, of their radii.

8. If two circles touch externally and a third circle is drawn to touch both of them internally, then the perimeter of the triangle formed by joining their centres is equal to a diameter of the third circle.

9 If two equal circles touch externally, prove that they intercept equal chords on any straight line drawn through the point of contact.

10. **If two circles touch internally or externally, and a straight line is drawn through the point of contact to cut both circles again, prove that**

(i) the tangents at the points of section are parallel;

(ii) the radii to the points of section are parallel

11. If two circles touch internally or externally at **A** and parallel tangents are drawn one to each, prove that the lines joining **A** to the points of contact of the tangents are either in the same straight line or at right angles to one another.

[*Hint.* Let **B** and **C** be the pts. of contact. Then in *internal* contact, **AB** and **AC** are one st. line, when **B** and **C** are on the *same* side of the line of centres and at rt. ∠s, when they are on *opposite* sides.]

12. **The straight lines joining the extremities of parallel diameters of two circles that touch pass through the point of contact.**

[Let **PP′**, **QQ′**, be parl diameters of ⊙s (**K**) and (**L**) which touch at **A**. Then in *internal* contact **PQ** and **P′Q′** pass thro. **A**, and in *external* contact **PQ′** and **P′Q** pass thro. **A**.]

13. Three circles touch one another externally at **A**, **B**, **C**; **AB**, **AC** are produced to cut the circle **BC** at **P** and **Q**; prove that **PQ** is a diameter of this circle and is parallel to the line of centres of the other two.

14. With a given point as centre, draw a circle to touch a given circle. How many such circles can be drawn?

15 **Find the locus of the centres of circles which touch a given circle at a given point.**

16. **Find the locus of the centres of circles of given radius which touch a given circle.**

17. A variable circle touches two fixed circles; prove that the sum or the difference of the distances of its centre from those of the fixed circles is constant (being equal to the sum or the difference of the radii of the fixed circles).

THEOREM 17. [Euc. III. 32.]

If a straight line touch a circle and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments of the circle.

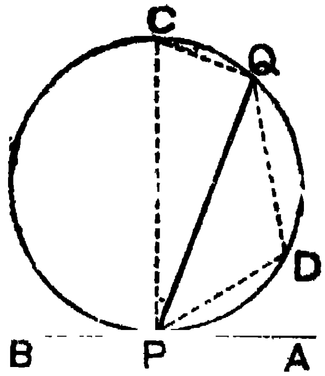


Fig. 1.

Let AB touch the circle PCD at P; and let PQ be a chord drawn from P, the point of contact.

To prove that

(i) *the* ∠APQ = *the angle in the alternate segment* PCQ;

(ii) *the* ∠BPQ = *the angle in the alternate segment* PDQ.

Construction. Through P, draw the diameter PC.

Take any point D in the arc PDQ; and join CQ, QD, DP.

Proof. (i) Because the ∠PQC in a semi-circle is a right angle; [III. 8.

∴ in the △PQC, the ∠CPQ + the ∠PCQ = a rt. angle.

But AB is a tangent and PC a diameter through the pt. of contact;

∴ the ∠APC = a rt. angle. [III. 13.

Hence the ∠APC = the ∠CPQ + the ∠PCQ.

Now take away the common ∠CPQ,

∴ the ∠APQ = the ∠PCQ, which is in the alternate segment.

(ii) Again, the ∠ BPQ = the supplement of the ∠ APQ
= the supplement of the ∠ PCQ.

But PDQC is a cyclic quadrilateral.

∴ the ∠ PDQ = the supplement of the ∠ PCQ.

∴ the ∠ BPQ = the ∠ PDQ, which is in the alternate segment.

Otherwise by the Method of Limits

Let AB touch the circle PCD at P, and let PQ be a chord from P, the point of contact.

Let C, D be any pts. in the arcs PCQ and PDQ.

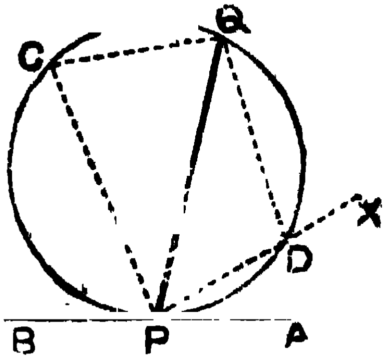


Fig. 2.

To prove that

the ∠ APQ = *the* ∠ PCQ *and the* ∠ BPQ = *the* ∠ PDQ.

Construction. Join PC, PD, QC, QD.
Produce PD to X.

Proof. Then because PCQD is a cyclic quadl.;

∴ the ∠ QDX = the ∠ PCQ; [III. 9, *Cor.*

and this is always true for any position of D however near to P.

Now suppose PX to turn about P, so that D moves along the arc PD towards P.

Thus when D moves up to P, and ultimately coincides with it, then PX becomes the tangent PA, QD coincides with QP and the ∠ QDX becomes the ∠ QPA.

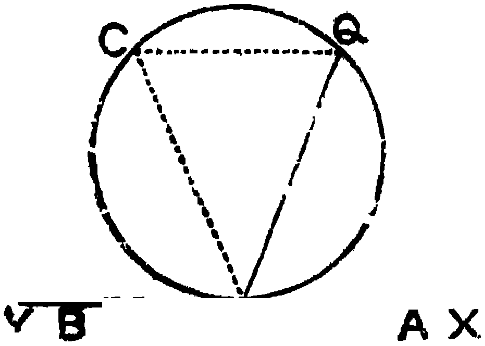
Hence the ∠ APQ = the ∠ PCQ, in the alternate segment.

And the ∠ BPQ = the supplement of the ∠ APQ
= the supplement of the ∠ PCQ
= the ∠ PDQ. Q. E. D.

* THEOREM 18.

[CONVERSE OF THEOREM 17.]

If through an extremity of a chord of a circle, a straight line is drawn making with the chord an angle equal to the angle in the alternate segment, then the straight line touches the circle.



Let PQ be a chord of a circle PCQ ; and let the st. line AB drawn through P make the $\angle APQ$ equal to the $\angle PCQ$ in the alternate segment.

To prove that AB *touches the circle at* P.

Construction. Let XY be the tangent to the circle at P.

Proof. Because XY is a tangent and PQ a chord,

∴ the $\angle XPQ$ = the $\angle PCQ$ in the alternate segment.

But the $\angle APQ$ = the $\angle PCQ$. [*Hyp.*

∴ the $\angle APQ$ = the $\angle XPQ$.

∴ AB coincides with XY, which is the tangent at P ;
that is, AB touches the circle at P.

Q.E.D.

EXERCISES LVII.

[*Numerical and Theoretical.*]

1. In Fig. 1 (Th. 17), if $\angle APD = 25°$ and $\angle CPQ = 20°$, find $\angle$s PDQ and PQD.

2. In Fig. 2 (Th. 17), if $\angle PQC = 60°$ and $\angle CPQ = 18°$, find $\angle$s APQ and PDQ.

3. In Fig. 2 (Th. 17), if $\angle$**QDX** = 70°, find $\angle$**APQ**.

4. **From a given circle, cut off a segment containg an angle equal to a given angle** [Euc. III. 34]

5. Divide a circle into two segments so that the angle in one may be double of the angle in the other.

6. If a tangent to a circle is parallel to a chord, the point of contact is the middle point of the arc cut off by the chord.

7. **If two circles touch one another, any straight line drawn through the point of contact cuts off similar segments from the circles.**

8. If two circles intersect, the angles subtended at the points of intersection by a common tangent are supplementary.

9. The perpendicular drawn from the middle point of an arc to the chord of the arc is equal to the perpendicular drawn from the same point to the tangent at an end of the chord.

10. A chord **AB** of a circle bisects the angle between the diameter through **A** and the perpendicular from **A** to the tangent at **B**.

11. Two circles intersect at **P** and **Q** ; **A** is a point on one of them ; **AP**, **AQ** cut the other circle in **B**, **C**. Prove that **BC** is parallel to the tangent at **A**.

12. **Two circles touch internally or externally and from the point of contact two straight lines are drawn to cut them ; prove that the lines joining the points of section are parallel.**

13. Two circles touch one another and from the point of contact straight lines are drawn to the extremities of any diameter of one of them ; prove that these lines meet the other circle at points which are the extremities of a parallel diameter.

14. Two circles touch internally and a straight line is drawn to cut them ; prove that the parts of it intercepted between the circles subtend equal angles at the point of contact.

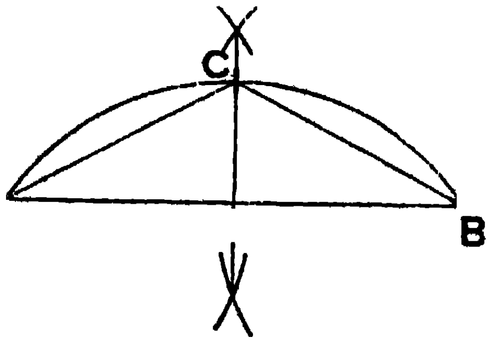
15. **Two circles touch internally and a chord of the greater is a tangent to the less ; prove that the chord is divided at its point of contact into parts which subtend equal angles at the point of contact.**

16. **AB** is a chord of a given circle ; another circle is described with centre **B** and radius **BA** cutting the given circle at **Q** and **AB** produced at **P** ; if in this circle arc **PQ** = arc **PT**, shew that **AT** is a tangent to the given circle.

17 **PQ** is a chord of a circle and **AB** a diameter perpendicular to it. Prove that **AP**, **BP** are the bisectors of the angles formed by **PQ** with the tangent at **P**.

CONSTRUCTIONS

Problem 1. To bisect an arc of a circle. [Euc. III. 30]



Let ACB be the given arc.

To bisect the arc ACB.

Construction. Join AB.

Bisect AB perpendicularly by OC, meeting the arc at C.

Then the arc ACB is bisected at C.

Proof. Join AC, BC.

Since OC is perpendicular bisector of AB, any point in OC is equidistant from A and B.

∴ AC = BC. [II. 7.

Now the equal chords AC, BC cut off equal arcs;

∴ the arc AC = the arc BC,

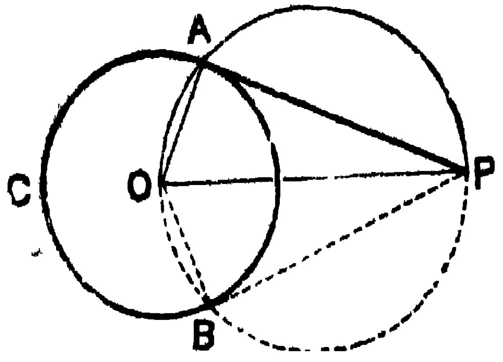
that is, the arc ACB is bisected at C.

EXERCISES LVIII.

1. Find the maximum area of a triangle, which can be inscribed in a given circle having a given chord for one of its sides.

2. Trisect a quadrant ($\frac{1}{4}$ part of the circumference) of a circle.

Problem 2. To draw a tangent to a given circle from a given external point.



Let ABC be the given circle whose centre is O, and P the given external point.

To draw a tangent to the circle ABC *from* P.

Construction. Join PO, and on OP as diameter describe a semi-circle OAP cutting the circle ABC at A.

Join PA.

Then PA is the tangent required.

Proof. Join OA.

The ∠OAP in a semi-circle is a right angle ;

∴ PA is perpendicular to the radius OA.

Hence PA is a tangent at A.

[III. 13, *Cor.* 1.

Note. If the circle **OAP** is completed, there is another semi-circle **OBP** on **OP**, which cuts the circle **ABC** again at **B**. Then if **PB** is joined, **PB** is another tangent from **P**, equal to **PA**. [I. 20].

Hence *from an external point, a* **pair of equal tangents** *can be drawn to a circle.*

EXERCISES LIX.

[*Practical and Theoretical.*]

1. Draw a circle of radius 1·5″, and from a point 1″ outside the circle, draw a tangent, and calculate its length.

2. From a point whose distance from the centre of a circle is 7 cm., draw a tangent. If the length of the tangent is 6·2 cm., calculate the radius of the circle.

3. A car, which has been moving at 15 miles per hour round a circular track of radius 470 ft. goes off at the same rate along a tangent; how far will it be from the centre after 5 sec.?

4. Draw a circle of radius 2·2 cm. and from an external point, draw a pair of tangents; if the distance of the chord of contact from the centre is 1 cm., find the distance of the point from the centre. Also find the length of a tangent.

5. **A** and **B** are two given points; through **A** draw a st. line at a distance of 3 cm. from **B**. How many such lines can be drawn?

6. Through a given point, draw a straight line at a given distance from another given point.

7. **Draw a tangent to a given circle from a given point on its circumference.**

[Draw the radius to that pt. and erect at the pt. a perpendicular to the radius. This perp. is the reqd. tangent (III. 13, *Cor.* 1), and this is the only tangent that can be drawn [III. 13, *Cor.* 3].

8. Draw a tangent to a given circle (i) parallel to, and (ii) making a given angle with, a given straight line.

9. Draw a pair of tangents to a given circle including a given angle.

10. Through a given point, draw a chord of a circle of radius 4·5 cm., at a distance of 3·2 cm. from the centre.

11. Through a given point, within or without] a circle, draw a chord at a given distance from the centre.

12. Through a given point, draw a chord of a circle of radius 3 cm., which shall be 3·5 cm. in length.

13. **Through a given point, within or without a circle, draw a chord of given length.**

14. In a given circle, draw a chord of given length (i) parallel to and (ii) making a given angle with, a given st. line. [See Exs. 8 and 13.]

Problem 3. To draw a common tangent to two given circles.

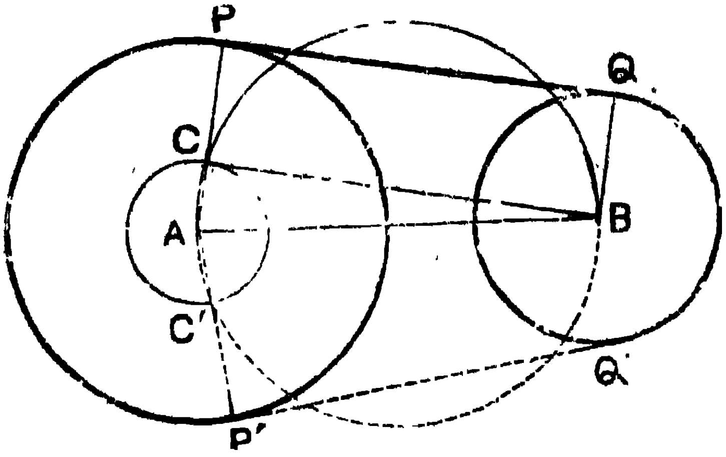


Fig. 1.

Let A and B be the centres and R, r the radii of the given circles ; and let R be greater than r.

To draw a common tangent to the circles (A) *and* (B).

Case I. [Direct Common Tangent.]

Construction. With centre A and radius $= R - r$ (*difference* of given radii), draw a circle.

From B, draw a tangent BC to this circle.

Join AC, and produce it to meet the circle (A) at P.

Through B, draw the radius BQ parl. to AP and in the *same* sense as AP.

Join PQ.

Then PQ is a common tangent to the given circles.

Proof. Now $AP = R$, and $AC = R - r$;

$$\therefore CP = R - (R - r) = r ;$$

$$\therefore CP = BQ.$$

Thus CP, BQ are equal and parallel ;

$\therefore$ BCPQ is a parallelogram.

Again BC is a tangent at C ; [*Const.*

$\therefore$ the $\angle$ BCP is a right angle.

$\therefore$ BCPQ is a rectangle ;

so that the angles at P and Q are right angles.

Hence PQ is a common tangent to the given circles.

Note 1. Since from B another tangent BC′ can be drawn to the construction circle at C′, P′Q′ will be another common tangent to the given circles.

Case II. [Transverse Common Tangent.]

Construction. With centre A and radius $= R + r$ (*sum* of given radii), draw a circle.

From B, draw a tangent BC to this circle.

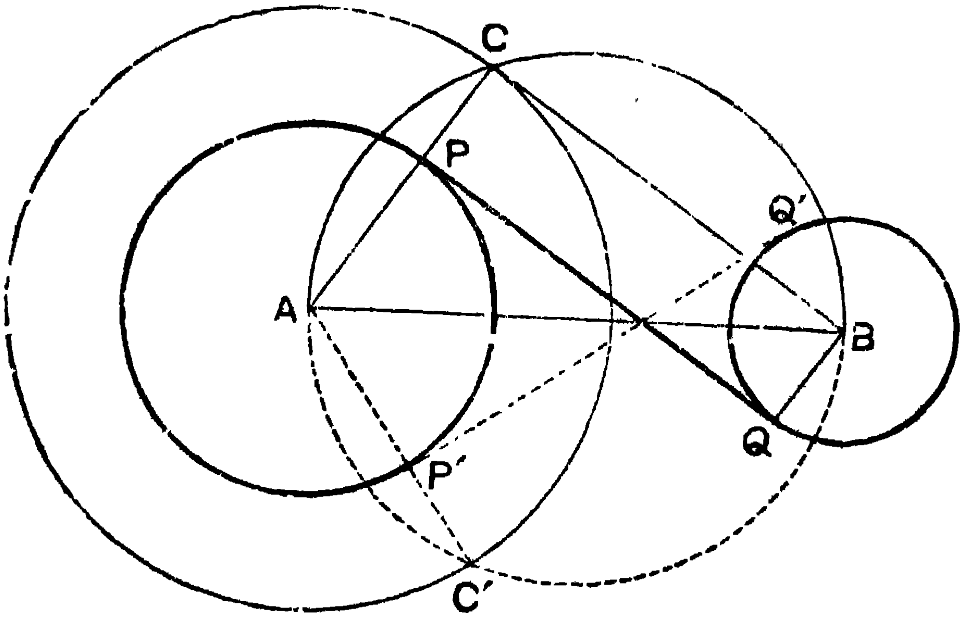


Fig. 2.

Join AC, meeting the circle (A) at P.

Through B, draw the radius BQ parl. to AP and in the *opposite* sense.

Join PQ.

Then PQ is a common tangent to the given circles.

Proof. Now $AP = R$, and $AC = R + r$;

$\therefore CP = (R + r) - R = r$;

$\therefore CP = BQ$.

Thus **CP**, **BQ** are equal and parallel ;

∴ **BCPQ** is a parallelogram.

Again **BC** is a tangent at **C** ;

∴ the ∠**BCP** is a right angle ;

∴ **BCPQ** is a rectangle ;

so that the angles at **P** and **Q** are right angles.

Hence **PQ** is a common tangent to the given circles.

Note. 1. From **B**, another tangent can be drawn to the construction circle at **C′**, so that **P′Q′** will be another common tangent to the given circles.

Note. 2. From Figs. 1 and 2, we see that there are **four common tangents** to two given circles when the circles are external to one another.

Def. When the points of contact of a common tangent are on the same side of the line of centres, it is called a **direct common tangent** ; and when the points of contact of a common tangent are on opposite sides of the line of centres, it is called a **transverse common tangent**.

Thus the two tangents in Fig. 1 are direct common tangents and the two in Fig. 2 are transverse common tangents.

EXERCISES LX.

[*Practical and Theoretical.*]

1. Draw two direct common tangents to two circles of radius 3 cm. each, when their centres are (i) 5 cm. apart, (ii) 7 cm. apart and (iii) 6 cm. apart.

2. Draw the four common tangents to two equal circles external to one another.

3. How many common tangents can be drawn to two equal circles, (i) when they are external to one another, (ii) when they touch externally, (iii) when they touch internally, (iv) when they intersect, and (v) when one is wholly inside the other but does not touch it?

Taking two circles of unequal radii, draw diagrams to illustrate each of these cases.

4. Draw two circles of radii 1·5 and 2 cm. with centres 4 cm. apart. Calculate the lengths of the four tangents and check by measurement.

5. The diameters of the fore and hind wheels of a carriage are 6 ft. and 7 ft. ; the distance between the points where they touch the road is 8 ft. ; find the distance between the centres of the wheels.

6. **Either pair of common tangents to two circles are equal.**

[Here "tangent" means the part of it between the points of contact.]

7. **Prove that each pair of common tangents to two circles intersect on the line of centres and make equal angles with it.**

8. **If two circles touch externally, either of the direct common tangents subtends a right angle at the point of contact.**

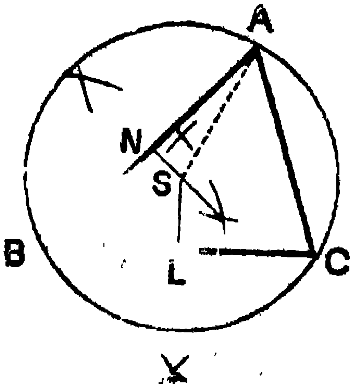
9. **P, Q** are the points of contact of a direct common tangent to two circles touching externally at **A**; shew that the circle described on **PQ** as diameter touches the line of centres at **A**.

10. Draw a straight line to cut two circles so that the intercepted chords may be of given lengths.

Problem 4. To circumscribe a circle about a given triangle

Let ABC be a traiangle.

To circumscribe a circle about the △ABC.



Construction. Bisect *any two* sides AB and BC at right angles by **NS** and **LS** meeting at **S**.

Then **S**, *the pt. of intersection*, is the centre of the required circle.

Proof. Now every point in **NS** is equidistant from A and B.

∴ SA = SB. [II. 7.

Similarly every pt. in LS is equidistant from B and C;

∴ SB = SC

∴ SA = SB = SC.

Hence if with centre S and radius SA, a circle is described, it will pass through B and C.

Thus ABC is the required circle.

Obs. This problem is the same as **to draw a circle to pass through three given points** (the vertices of a triangle.)

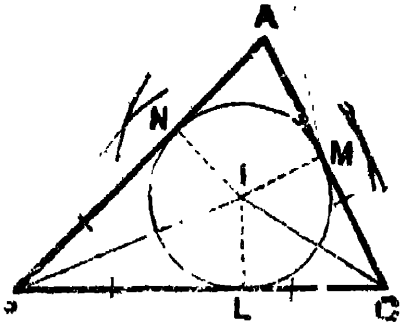
Def. The circle circumscribed about a triangle is called the **circumcircle** of the triangle, its centre is called the **circumcentre**, and its radius the **circumradius**.

The circumcentre is usually represented by the letter S.

Note. From this Problem, it is evident that **to find the circumcentre** of a triangle, we **bisect any two sides** at right angles; for the perpendicular bisectors of the *three* sides are concurrent, and *the point of concurrence is the circumcentre.*

Question. Draw the diagram to illustrate the *three* cases when the triangle is (i) acute-angled, (ii) right-angled, and (iii) obtuse-angled. Where does the circumcentre lie in each of the three cases?

Problem 5. To inscribe a circle in a given triangle.



Let ABC be a triangle.

To inscribe a circle in the △ ABC.

Construction. Bisect *any two* angles B and C by BI and CI meeting at I.

Then I, *the pt. of intersection,* is the centre of the inscribed circle.

Proof. Draw IL, IM, IN perpendicular to BC, CA and AB.

Now every point in BI is equidistant from BC, BA;

∴ IL = IN. [II. 8.

Also every point in CI is equidistant from CB, CA;

∴ IL = IM.

∴ IL = IM = IN.

Hence the circle with centre I and radius IL will pass through M and N.

And this circle will touch BC, CA and AB, for the angles at L, M and N are rt. angles. [III. 13, *Cor.* 1.

Thus LMN is the circle inscribed in the △ ABC.

Def. The circle inscribed in a triangle is called the **incircle** of the triangle, its centre is called the **incentre** and its radius the **inradius.**

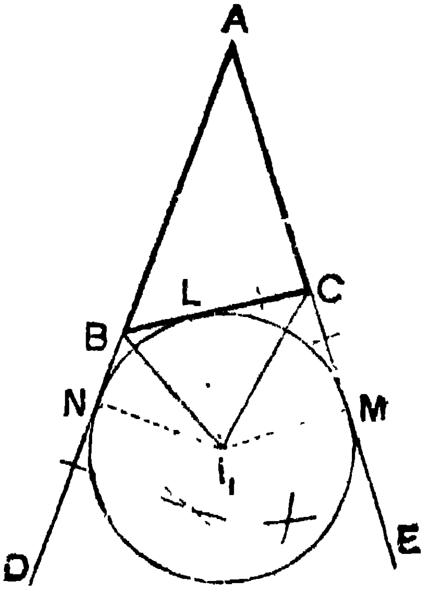
It is usual to represent the incentre by the letter I.

Note. From this Problem we see that **to find the incentre** of a triangle, we **bisect any two angles,** since the bisectors of the *three* angles are concurrent, and *the point of concurrence is the incentre.*

Rider. If AI produced meets BC in D, shew that
the ∠BID = the ∠CIL.

Def. The circle which touches one side of a triangle and the other two sides produced is called an **escribed circle** of the triangle, its centre an **ex-centre** and its radius an **ex-radius.**

Problem 6. To draw an escribed circle of a given triangle.



Let ABC be a triangle.

To draw a circle to touch BC *and* AB, AC *produced.*

Construction. Bisect the exterior angles at B and C by BI_1 and CI_1 meeting at I_1.

Then I_1, *the pt. of intersection,* is the centre of the required circle.

Proof. Draw I_1L perp. to BC and I_1M, I_1N to AC and AB produced.

Now every point in B_1 is equidistant from BC, BD ;

$$\therefore I_1L = I_1N. \qquad [\text{II. } 8.$$

Also every point in CI_1 is equidistant from CB, CE ;

$$\therefore I_1L = I_1M.$$

$$\therefore I_1L = I_1M = I_1N.$$

Hence the circle with centre I_1 and radius I_1L will pass through M and N ;

And this circle will touch BC, CE and BD, for the angles at L, M and N are right angles.

Thus LMN is an escribed circle of the △ABC.

Note. From this Problem it is easy to see that **to find the excentre** corresponding to any side, we **bisect the exterior angles** adjacent to that side, the point of intersection of the bisectors being the excentre.

Similar construction will enable the student to draw **two** other circles to touch respectively the sides AC and AB and the corresponding two sides produced. Thus there are **three** escribed circles of a triangle but only **one** inscribed circle.

Obs. From Problems 5 and 6, we see how **to draw a circle to touch three given straight lines.**

Hence this problem has *four* solutions.

Def. The circle **LMN** is called the **escribed** circle of the Δ ABC, *corresponding to the side* a, *or opposite to the angle* A.

EXERCISES LXI.

[*Theoretical.*]

N. B.—In the following exercises, a, b, c, denote the sides opposite to the angles **A**, **B**, **C** of a Δ**ABC**; **I** the incentre; I_1, I_2, I_3 the excentres corresponding to a, b, c; r the inradius; r_1, r_2, r_3, the exradii corresponding to a, b, c; and $2s=a+b+c$.

1. Any circle whose centre is **I** cuts off equal chords from the sides of the Δ**ABC**.

2. Prove that **A**, **I** and I_1, are collinear.

3. If the incentre and the circumcentre coincide, the triangle is equilateral.

4. If the join of the incentre and the circumcentre of a triangle passes through an angular point, the triangle is isosceles.

5. If **AI**, **BI**, **CI** cut the incircle at **X**, **Y**, **Z**, shew that **X**, **Y** and **Z** are respectively the incentres of the Δs **AMN**, **BNL** and **CLM**. [See Fig. Prob. 5.]

6. The diagonals of a given quadrilateral **ABCD** intersect at **O**; shew that the circumcentres of the Δs **OAB**, **OBC**, **OCD** and **ODA** are the angular points of a parallelogram.

7. The inradius of a right-angled triangle is half the difference between the sum of the sides and the hypotenuse.

8. **Draw a circle to touch three given straight lines which are neither concurrent nor all parallel.**

9. Inscribe a circle in a given sector of a circle.

10. Draw a circle to touch a given circle and the tangents to the circle at two given points.

11 and 12. In the figure where (I) is the inscribed ⊙ and (I_1) an escribed ⊙ touching a, shew that

(i) $AM+CL+BN$
$=AN+BL+CM=s$.

(ii) $AM=AN=s-a$.

(iii) $BL=BN=s-b$.

(iv) $CL=CM=s-c$.

(v) $AM+AN=b+c-a$.

(vi) $AQ=AR=s$.

(vii) $CQ=CP=s-b$.

(viii) $BR=BP=s-c$.

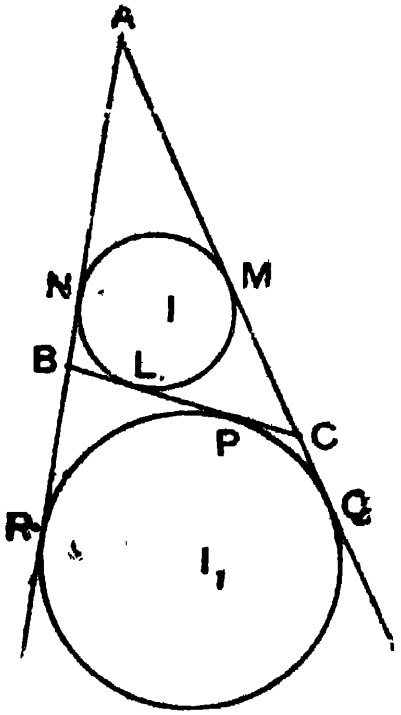
(ix) $MQ=NR=a$.

Also (x) $CL=BP$.

(For $CL=s-c=BP$.)

(xi) $CP=BL$.

(For $CP=s-b=BL$.)



These relations should be ***very carefully*** remembered.

13. The area of any triangle$=rs=r_1(s-a)=r_2(s-b)=r_3(s-c)$.

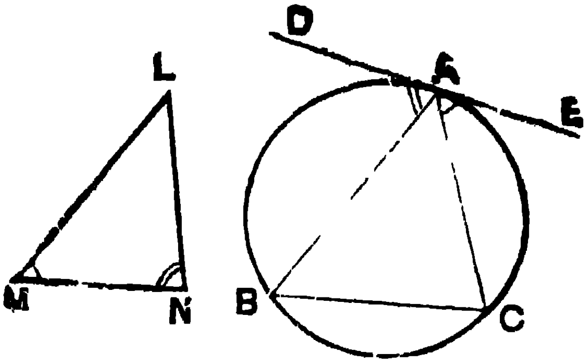
14. Construct a triangle having given the base, the altitude and the circumradius.

15. I is the incentre of a triangle **ABC** ; **AI** is produced to cut the circumcircle at **O** ; prove that $OB=OI=OC$.

16. Prove that the circumcentre of the $\triangle BI_1C$ [Fig. Problem 6] lies on the circumcircle of the $\triangle ABC$.

CONSTRUCTION OF REGULAR FIGURES IN OR ABOUT A CIRCLE

Problem 7. In a given circle, to inscribe a triangle equiangular to a given triangle. [EUC. IV. 2.]



Let ABC be the given circle and LMN the given triangle.

Construction. At any pt. A on the circumference of the ⊙ ABC, draw the tangent DAE.

At A, make the ∠EAC = the ∠M

and the ∠DAB = the ∠N

Join BC.

Then ABC is the required triangle.

Proof. Now the ∠M = the ∠EAC = the ∠ABC in the alt. segment; and the ∠N = the ∠DAB = the ∠ACB in the alt. segment. [III. 17.

∴ the ∠L = the ∠BAC.

ALTERNATIVE METHOD

Construction. Find O the centre of the given circle. [Draw the necessary Figure.]

Take any pt. A on the circumference.

Make the ∠AOC = 2∠M and the ∠AOB = 2∠N.

Join AB, BC, CA.

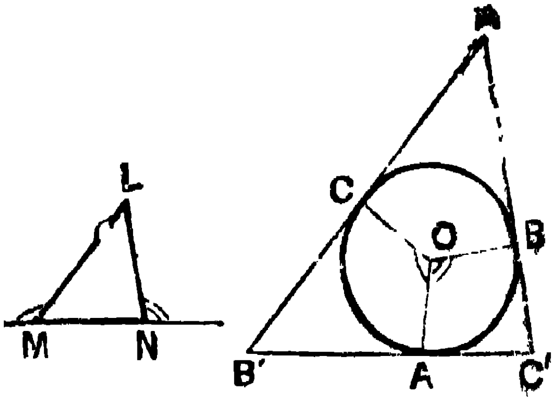
Then ABC is the required triangle.

Proof. Now the ∠AOC at the centre = 2∠B = 2∠M [*Const.*

∴ the ∠B = the ∠M.

Similarly, the ∠C = the ∠N. ∴ the ∠A = the ∠L.

Problem 8. About a given circle, to circumscribe a triangle equiangular to a given triangle. [Euc. IV. 3].



Let ABC be the given circle and LMN the given triangle.

Construction. Produce MN both ways. Draw any radius OA.

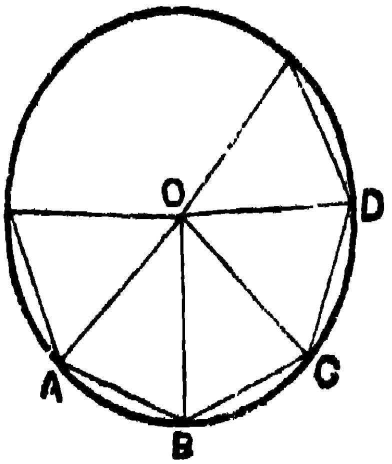
At O, make the ∠AOC = supplement of the ∠LMN and the ∠AOB = supplement of the ∠LNM.

Draw A′B′ perp. to OC at C, B′C′ to OA at A and C′A′ to OB at B.

Then A′B′C′ is the required triangle.

Proof. The ∠LMN = suppt. of the ∠AOC = the ∠B′;
and the ∠LNM = suppt. of the ∠AOB = the ∠C′,
∴ the ∠ L = the ∠A′.

Problem 9. To inscribe a regular n-gon in a given circle.



Let O be the centre of the given ⊙.

Since the angle at O is 4 rt. ∠s, it is evident that each side of the inscribed figure will subtend at O an angle $= \frac{360^\circ}{n}$.

Construction. Draw any radius OA.

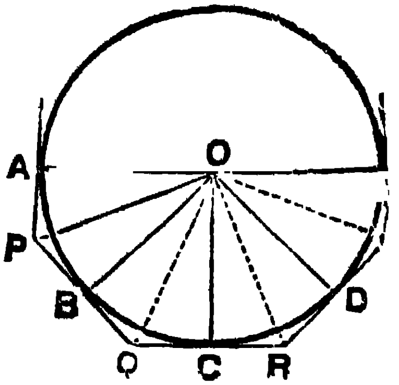
Make the ∠AOB = the ∠BOC = the ∠COD = ... = $\frac{360^\circ}{n}$, thus dividing the circumference into n equal arcs at A, B, C, D,.........

Draw chords AB, BC, CD,......

Thus ABCD......is the required polygon.

[The construction may be proved by shewing that △s **AOB**, **BOC**, **COD**,............are congruent. Hence the polygon **ABCD**............is both equilateral and equiangular.]

Problem 10. To circumscribe a regular n-gon about a given circle.



Construction. Make as in Problem 9, $\angle AOB = \angle BOC = \angle COD = \ldots\ldots = \dfrac{360^\circ}{n}$, thus dividing the circumference into n equal arcs at A, B, C, D,......

Draw tangents at A, B, C, D,......., intersecting at P, Q, R, S,.........

Then PQRS......is the required polygon.

[Join OP, OQ, OR, OS,..... Then △s OPA, OPB, OQB, OCQ,......are all congruent; so also △s OPQ, OQR, ORS,......... are all congruent. Hence the polygon PQRS.......is both equilateral and equiangular.]

Cor. *If the circumference of a circle be divided into* n *equal arcs, then* (i) *the chords of these arcs will form a regular* n*-gon inscribed in the circle*; *and* (ii) *if tangents be drawn at the points of division, they will form a regular* n*-gon circumscribed about the circle*; or in other words, if a regular polygon of n sides be inscribed in a circle, a regular polygon of the *same number* of sides can be circumscribed about the circle.

Note.

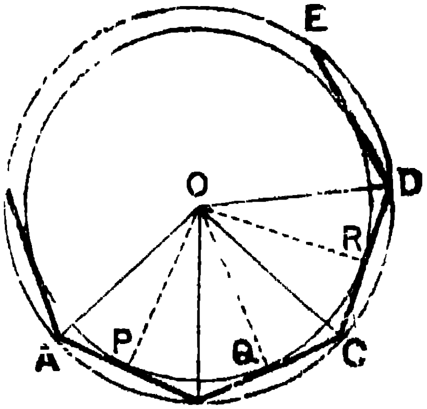
In a given circle, **to inscribe**

(i) a regular **5-gon**. Make $\angle AOB = \angle BOC = \ldots\ldots = 72^\circ$.
[This may be done as above, or as in IV. Prob. 8.]
(ii) a regular **6-gon**. Make $\angle AOB = \angle BOC = \ldots\ldots = 60^\circ$.
(iii) a regular **8-gon**. Make $\angle AOB = \angle BOC = \ldots\ldots = 45^\circ$.

To circumscribe about a given circle a regular **5-gon, 6-gon,** or **8-gon,** draw tangents at the pts. of division in (i), (ii) or (iii) respectively.

Problem 11. To inscribe or circumscribe a circle in or about a regular n-gon.

Let **ABCD**...be a regular *n*-gon.



Construction. Bisect any two consecutive $\angle^s$ **ABC, BCD** by **BO, CO** intersecting at **O.**

Then **O** is the centre both of the **inscribed** and **circumscribed circles.**

[Join **OD** and prove that $\triangle$s **OCB, OCD** are congruent ; thus **OD** bisects $\angle$**CDE.** Similarly shew that the bisectors of all the interior $\angle$s meet at **O.** Also $\because$ **OB, OC, OD,**.. .. are bisectors of equal $\angle$s at **B, C, D,**......., $\therefore$ **OB = OC = OD =** ... ..Hence **O** is the centre of the **circumscribed** ⊙

Again draw perps. **OP, OQ, OR,**......and prove that $\triangle$s **OBP, OBQ, OCQ** .....are congruent ; thus **OP = OQ = OR =** ...Hence **O** is the centre of the **inscribed** ⊙.]

EXERCISES LXII.

[*Practical and Theoretical.*]

1. In a circle of radius 1·5″, inscribe a triangle two of whose angles are 30° and 50°.

2. About a circle of radius 2 cm., circumscribe a triangle two of whose angles are 40° and 60°.

3. In a circle of radius 3 cm., inscribe an isosceles triangle each of whose angles at the base is double of the vertical angle.

4. Circumscribe about a circle of radius 2·8 cm. a right-angled triangle, an acute angle of which is 48°.

5. Inscribe an equilateral triangle in a given circle and another about it. Shew that the area of the first triangle is a fourth of that of the second.

If the radius of the circle is 4 cm., find the area of each triangle.

6 A square of side 1·8″ circumscribes a circle ; find its radius.

If a square is inscribed in the circle, find its side.

7. A square is inscribed in a circle and another is circumscribed about it ; prove that the area of the first is half that of the second.

8. By inscribing an equilateral triangle in a given circle, shew how to inscribe in it a regular hexagon.

9. In a given circle, inscribe (i) a square, (ii) a regular pentagon, and (iii) a regular hexagon.

10 **A side of a regular hexagon inscribed in a circle is equal to a radius of the circle.**

11. About a given circle, circumscribe (i) a square, (ii) a regular pentagon and (iii) a regular hexagon.

12. Inscribe a circle in (i) a square, (ii) a regular pentagon, and (iii) a regular hexagon.

13. Circumscribe a circle about (i) a square, (ii) a regular pentagon, and (iii) a regular hexagon.

14. Inscribe and circumscribe a regular octagon in and about a circle.

15. In a circle of radius 3·2 cm., inscribe a rectangle, one of whose sides is 5 cm.

16. Prove that of all rectangles inscribed in a circle, the square has the maximum area.

17. Circumscribe about a given circle a triangle having its sides parallel to three given straight lines.

THE CIRCUMFERENCE OF A CIRCLE

It has been ascertained that the length of the circumference of a circle is roughly 3 times the length of its diameter; and this is the case *whatever be the size of the circle.* Hence we conclude that the ratio of the circumference of a circle to its diameter is *constant.*

This constant ratio is invariably denoted by the Greek letter π (pronounced *Pi*), so that

$$\frac{\text{circumference}}{\text{diameter}} = \pi \quad \textit{i.e.,} \text{ circumference} = \pi \times \text{diameter}.$$

We have said that the value of π is roughly 3; but its value cannot be found *exactly*; hence the circumference of a circle and its diameter are two *incommensurable* quantities.

The value of π can however be calculated to any degree of approximation that may be necessary.

Thus $\pi = \frac{22}{7}$ or $\frac{355}{113}$ or $3{\cdot}14159$ or $3{\cdot}14159265$, &c.;

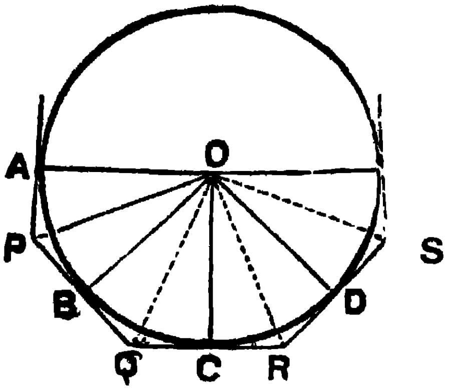
each successive value being a nearer approximation to the true value.

Hence if r denotes the radius of a circle,

$$\textbf{Circumference} = \pi \times \text{diameter} = \mathbf{2\pi r}.$$

THE AREA OF A CIRCLE

Let PQ, QR, RS……be the sides of a regular polygon of n sides circumscribed about a circle of centre O and OB, OC, OD………the radii drawn to the pts. of contact, so that OB, OC, OD,………are respectively perp. to PQ, QR, RS,………



Thus if r = the radius of the circle, then

$$r = OB = OC = OD = \ldots\ldots ; \text{ and}$$

Area of polygon PQRS…

$$= \triangle OPQ + \triangle OQR + \triangle ORS + \ldots\ldots$$
$$= \tfrac{1}{2}OB.PQ + \tfrac{1}{2}OC.QR + \tfrac{1}{2}OD.RS + \ldots\ldots$$
$$= \tfrac{1}{2}r.PQ + \tfrac{1}{2}r.QR + \tfrac{1}{2}r.RS + \ldots\ldots$$
$$= \tfrac{1}{2}r(PQ + QR + RS + \ldots\ldots)$$
$$= \tfrac{1}{2}r \times \text{perimeter of polygon.}$$

Now if n is increased without limit, the perimeter of the polygon approaches nearer and nearer to the circumference of the circle, and ultimately equals the circumference ; and then also the area of the polygon equals the area of the circle. Hence

$$\textbf{Area of circle} = \tfrac{1}{2}r \times \textbf{circumference}$$
$$= \tfrac{1}{2}r.2\pi r$$
$$= \pi r^2.$$

THE AREA OF A SECTOR OF A CIRCLE

The angle at the centre of a $\odot = 4$ right angles or 360°.

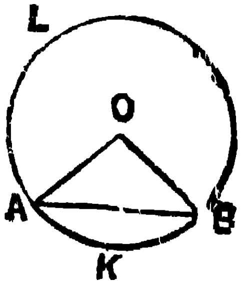
If this angle be divided into 360 equal angles by the radii of the circle, then any *two* consecutive radii will contain an angle of 1° and cut off an arc whose length $= \frac{1}{360}$ of the circumference. Also the area of the circle will be divided into 360 equal sectors of angle 1° each; and the area of each of these sectors $= \frac{1}{360}$ of the area of the circle.

If therefore the $\angle$ AOB of the sector OAKB contain x°,

$$\text{the arc AKB} = \frac{x}{360} \times \text{circumference}$$

and the sector $\text{OAKB} = \frac{x}{360} \times$ area of circles. Hence

$$\text{Area of sector OAKB} = \frac{x}{360} \times \text{area of circle}$$



$$= \frac{x}{360} \times \tfrac{1}{2}r \times \text{circumference}$$

$$= \frac{x}{360} \times \text{circumference} \times \tfrac{1}{2}r$$

$$= \text{arc AB} \times \tfrac{1}{2}r$$

$$= \tfrac{1}{2}\,\text{arc AB} \times \text{radius.}$$

That is, **the area of a sector of a circle is half the product of the arc of the sector into the radius of the circle.**

THE AREA OF A SEGMENT OF A CIRCLE

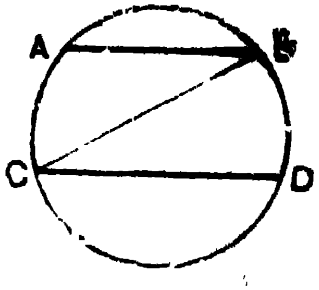
From the above, it is easy to find the area of the **minor segment** ABK. For

Area of segment ABK
= *area of sector* OAKB − *area of* △OAB.

Hence the area of **major segment** ABL
= area of circle − area of minor segment ABK.

APPENDIX III

1. THEOREM. *Parallel chords of a circle intercept equal arcs*; and conversely, *chords of a circle which intercept equal arcs are parallel.*



Let **ABDC** be a circle.

(i) Let **AB** ∥ **CD**.

Then arc **AC** shall be = arc **BD**.

Join **BC**.

Then ∠ **ABC** = alt. ∠**BCD** ;

∴ arc **AC** = arc **BD**.

(ii) Let arc **AC** = arc **BD**.

Then **AB** shall be ∥ **CD**.

Now arc **AC** = arc **BD** ; ∴ ∠ **ABC** = ∠**BCD**, and they are alt. ∠s, ∴ **AB** ∥ **CD**.

Cor. 1. *The arc* ACD = *the arc* BDC ; *and the* ∠ABD = *the* ∠BAC.

Cor. 2. *If two angles at the circumference of a circle are supplementary, the sum of the arcs on which they stand is equal to the whole circumference.*

Ex. 1. **AB**, **AC** are chords of a circle parallel respectively to two other chords **A′B′** and **A′C′** ; prove that **BC′** is parallel to **B′C**.

Ex. 2 **Any two parallel straight lines, drawn through the points of intersection of two circles and terminated by them, are equal.**

2. THEOREM. *If two chords of a circle intersect* within *or* without *it, the angle between them is equal to the angle at the circumference standing on an arc equal to the* sum *or* difference *of the intercepted arcs.*

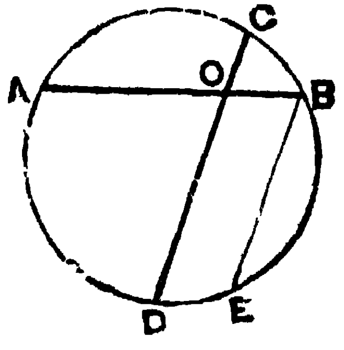


Fig. 1.

Let **AB**, **CD** be two chords of a circle.

First, let them intersect at **O** within the circle [Fig. 1.]

Then the ∠**AOD** = the angle at the circumference which stands on the *sum* of the arcs **AD**, **CB**.

Draw **BE** ∥ **CD** Then arc **CB** = arc **DE**, and ∴ arc **AE** = sum of arcs **AD**, **CB**.

And ∠ **AOD** = ∠**ABE** which stands on arc **AE**.

Similarly, it may be shewn that the ∠AOC = the angle at the circumference which stands on the sum of the arcs AC, BD.

Secondly, let AB, CD intersect at O without the circle [Fig. 2.]

Then the ∠AOD = the angle at the circumference which stands on the *difference* of the arcs AD, CB.

Draw BE ∥ CD. Then arc CB = arc DE, and ∴ arc AE = difference of arcs AD, CB.

Now ∠AOD = ∠ABE which stands on arc AE.

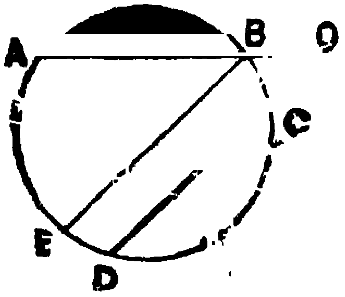


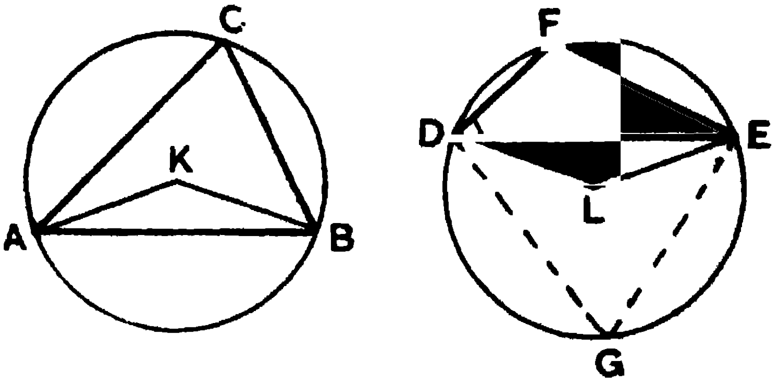
Fig 2.

Cor. *If two chords of a circle intersect at right angles, the sum of the opposite arcs intercepted by them is a semi-circumference* ; and *conversely.*

Note. Evidently ∠AOD = ½ the angle at the *centre* standing on an arc equal to the *sum* or *difference* of the intercepted arcs.

3. THEOREM. *Two triangles which have their bases equal and vertical angles equal or supplementary have equal circum-circles.*

It is enough to prove the case in which the vertical angles are supplementary.



Let ABC, DEF be two △s, whose bases AB, DE are equal and the vertical ∠s ACB, DFE supplementary.

Let ABC, DFE be the circum-circles of the △s.

Find the centres K and L ; take any pt. G in the arc DGE ; join DG, EG, KA, KB, LD and LE.

Now the $\angle ACB =$ suppt. of the $\angle DFE =$ the $\angle DGE$;

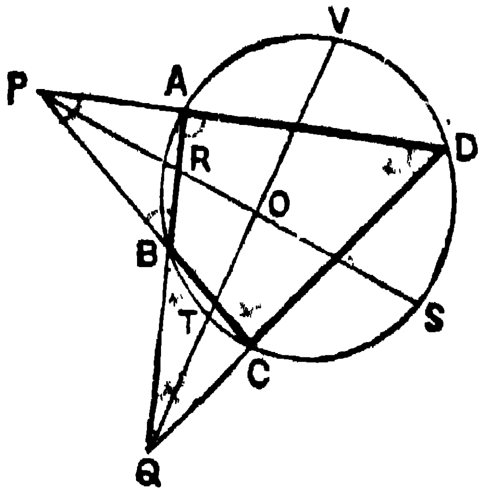
$\therefore$ the $\angle K = 2\angle ACB = 2\angle DGE =$ the $\angle L$.

But the $\angle K + 2\angle KAB = 2$ rt. $\angle$s $=$ the $\angle L + 2\angle LDE$;

$\therefore$ the $\angle KAB =$ the $\angle LDE$.

Hence the $\triangle$s **KAB, LDE** are congruent; so that $KA = LD$; thus the radii of the circles being equal, the circles are equal.

4. THEOREM. *If the opposite sides of a cyclic quadrilateral be produced to meet, then the bisectors of the angles thus formed intersect at right angles.*



Let the opp. sides of the cyclic quadl. **ABCD** be produced to meet at **P** and **Q**. Let the bisectors of the $\angle$s at **P** and **Q** meet at **O**.

To prove that the angle **POQ** is a right angle.

The $\angle BAD =$ the $\angle P +$ the $\angle ABP =$ the $\angle P +$ the $\angle D$.

and the $\angle BCD =$ the $\angle Q +$ the $\angle CBQ =$ the $\angle Q +$ the $\angle D$.

$\therefore$ the $\angle P +$ the $\angle Q + 2\angle D =$ the $\angle BAD +$ the $\angle BCD = 2$ rt. $\angle$s;

$\therefore \frac{1}{2}\angle P + \frac{1}{2}\angle Q +$ the $\angle D = 1$ rt. angle.

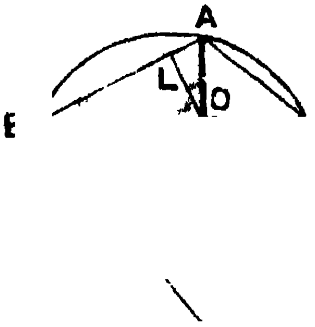
But the $\angle POV =$ the $\angle OPT +$ the $\angle OTP$

$=$ the $\angle OPT +$ the $\angle BQT +$ the $\angle QBT$

$= \frac{1}{2}\angle P + \frac{1}{2}\angle Q +$ the $\angle D = 1$ rt. angle.

$\therefore$ the angle at **O** is a rt. angle.

5. THEOREM. *If the diagonals of a cyclic quadrilateral are at right angles, the perpendicular from the point of intersection to any side when produced bisects the opposite side.* (BRAHMAGUPTA'S THEOREM.)



Let **AC**, **BD** be at rt. angles and **OL** perp. to **AB**; and let **LO** produced meet **DC** at **M**.

To prove that **LO** bisects **DC** at **M**.

$$\angle COM = \angle AOL$$
$$= \text{compt. of } \angle LOB$$
$$= \angle OBL \text{ or } ABD$$
$$= \angle ACD \text{ or } OCM$$
$$\therefore OM = MC.$$

$\therefore$ **M** is the mid pt. of the hypot. **CD**

The Converse. *If the diagonals of a cyclic quadrilatral are at right angles, the join of the middle point of a side to the point of intersection of the diagonals when produced is perpendicular to the opposite side.*

[Supply Proof.]

Ex. 3. **ABCD** is a cyclic quadrilateral whose diagonals **AC**, **BD** intersect at rt. angles in **O**; **AX** is drawn perp. to **BC** (produced, if necessary); prove that **DX** is bisected by the perp. (produced, if necessary) from **O** to **BC**.

Ex. 4. The diagonals of a cyclic quadrilateral are at rt. angles; prove that the distance of the centre of the circum-circle from a side is half the opposite side.

Ex. 5. The diagonals of a cyclic quadrilateral are at rt. angles; prove that the join of the intersection of the diagonals to the centre of the circum-circle is bisected by the join of the mid points of the opposite sides.

Ex. 6. If the diagonals of a cyclic quadrilateral are at rt. angles, prove that the eight points, *viz.*, the feet of the perps. from the intersection of the diagonals to the sides and the mid pts. of the sides, are concyclic.

ORTHO-CENTRE AND PEDAL TRIANGLE

Def. The point of concurrence of the **altitudes** of a triangle is called its **orthocentre** ; and the triangle formed by joining the feet of the altitudes is called the **orthocentric** or **pedal triangle** with respect to the original triangle. Thus **O** is the orthocentre and the Δ **PQR** [fig. of (7)] is the pedal triangle, of the Δ **ABC**.

6. THEOREM. *The three altitudes of a triangle are concurrent.*

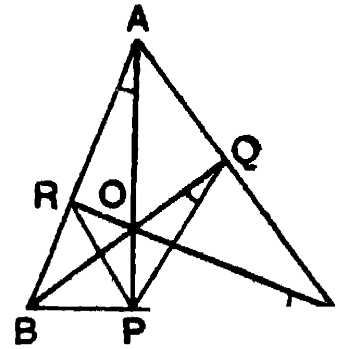
We have already given a proof of this Theorem. [App. I (6).] The following is another proof.

Draw **AP**, **BQ** perp. to **BC** and **AC**, to intersect at **O**.

Join **CO** and produce it to meet **AB** at **R**.

To prove that **CR** is perp. to **AB**.

Join **PQ**, **PR**.



Because the angles at **P** and **Q** are rt. angles,

∴ **ABPQ** and **OPCQ** are cyclic quadls.

∴ the ∠**BAP** = the ∠**BQP** or **OQP** = the ∠**OCP**.

Thus **PR** subtends equal angles at **A** and **C** ;

∴ the pts. **A**, **C**, **P**, **R** are concyclic [III. 7].

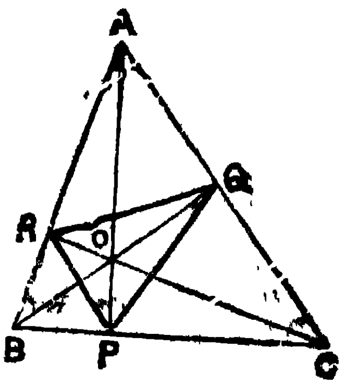
Hence the ∠**ARC** = the ∠**APC** in the same segment
= a rt. angle.

The letter **O** usually represents the orthocentre of a triangle.

Question 1. What is the orthocentre of a right-angled triangle? When does the orthocentre fall without the triangle ? When within ?

Question 2. What is the orthocentre of the Δ**BOC** ? What of the Δ**AOC** ? What of the Δ**AOB**

7. THEOREM. *Each altitude of an acute-angled triangle bisects the angle of the pedal triangle through which it passes.*



Let **ABC** be an acute-angled △; and **AP, BQ, CR** the altitudes meeting at **O**, so that **PQR** is the pedal triangle.

To prove that **AP, BQ, CR** bisect the ∠s **QPR, PQR** and **PRQ** respectively.

The angles at **P, Q** and **R** are rt. angles; ∴ **OPBR, BCQR** and **OPCQ** are cyclic quadls.

Hence the ∠**OPR**=the ∠**OBR** or **QBR**=the ∠**RCQ** or **OCQ**=the ∠**OPQ**; that is, **AP** bisects the ∠**QPR**.

Similarly it may be shewn that **BQ, CR** bisect the ∠s **PQR** and **PRQ**.

Cor. O is the incentre of the pedal triangle.

Note. Each of the △s **ABC, AOB, BOC** and **AOC** has the *same* pedal △, *viz.*, **PQR**. Hence two of the angles of the pedal △ of the obtuse-angled △**AOB** are bisected by the sides **AO, BO** which contain the *obtuse* angle. Similar remark holds good with respect to the obtuse-angled △s **BOC** and **AOC**.

Ex. 7. Of the four points **A, B, C, O,** each is the orthocentre of the triangle formed by joining the other three.

Ex. 8. The angle subtended by a side of a triangle at the orthocentre is supplementary to the angle opposite that side.

Ex. 9. **Prove that the circles circumscribed about the triangles ABC, BOC, COA and AOB are all equal.**

Ex. 10. **Each pair of sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet.**

Let **ABC** be a △, and **PQR** its pedal △ [see Fig. of (7)].

First take the pair of sides **PQ** and **PR**.

To prove that the ∠**CPQ**=the ∠**BPR**.

Now **ABPQ** is a cyclic quadl.; ∴ the ∠**CPQ**=the ∠**A**,

Also **ACPR** is a cyclic quadl.; ∴ the ∠**BPR**=the ∠**A**.

∴ the ∠**CPQ**=the ∠**BPR**.

Similarly it may be shewn that the ∠**CQP**=the ∠**AQR**.

and the ∠**ARQ**=the ∠**BRP**.

Obs. The sides of the △**ABC** are the external bisectors of the angles of the pedal △, so that the vertices **A, B** and **C** are its ex-centres.

Ex. 10a. Any triangle, inscribed in a given triangle and having each pair of its sides equally inclined to the sides of the given triangle, must be the pedal triangle.

Let **PQR** be a $\triangle$ inscribed in the $\triangle$ **ABC** [see Fig. of (7)], such that its sides **PQ**, **PR** are equally inclined to **BC** ; **QP**, **QR** to **AC**, and **RQ**, **RP** to **AB**.

To prove that the $\triangle$**PQR** must be the pedal $\triangle$.

Produce **RP** to **D**. Then the $\angle$**CPQ**$=$the $\angle$**BPR**$=$the $\angle$**DPC** ; thus **PC** or **BC** is the external bisector of the $\angle$**P** of the $\triangle$ **PQR**. Similarly **AC** and **AB** are the external bisectors of the $\angle$s **Q** and **R**.

Now the external bisectors of the $\angle$s **Q** and **R** meet at **A** ;

$\therefore$ **AP** is the internal bisector of the third $\angle$**P** [*App.* I. (5)].

Hence the $\angle$**APC**, being between the two bisectors of the $\angle$**P**, is a rt. $\angle$; *i.e.*, **AP** is perp. to **BC**. Similarly **BQ** is perp. to **AC** and **CR** to **AB**. Thus the $\triangle$**PQR** is the pedal $\triangle$.

It is thus seen that there is *only one inscribed triangle* having this peculiar property.

Ex. 11. The triangles **AQR**, **PBR**, **PQC** are equiangular to one another and to the triangle **ABC**.

Ex. 12. **ABC** is a triangle ; **I** is its in-centre, and I_1, I_2, I_3 are its ex-centres corresponding to the sides **BC**, **CA** and **AB**. Prove that

(i) The pts. I_1, **C**, I_2 are collinear ; also are the pts.

I_2, **A**, I_3 and I_3, **B**, I_1.

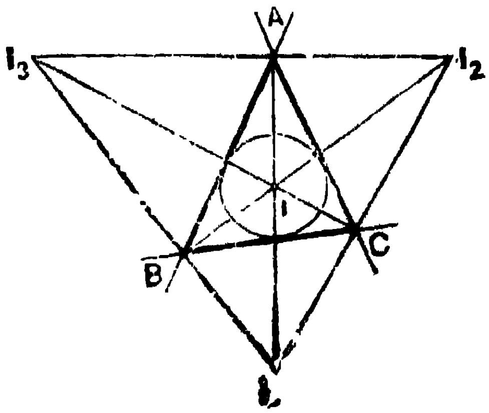
(ii) The pts. **A**, **I**, I_1, are collinear ; so also are the pts.

B, I, I_2 and **C**, I, I_3.

(iii) $AI I_1$, $BI I_2$, $CI I_3$ are the altitudes of $\triangle$ $I_1 I_2 I_3$,

and

(iv) **I** is the orthocentre of $\triangle I_1 I_2 I_3$.



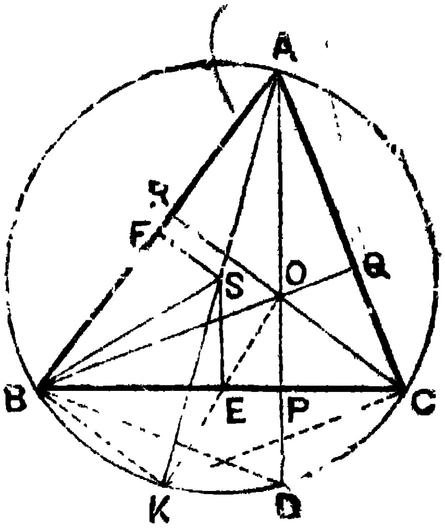
Ex. 13. Prove that the triangle formed by joining the points of contact of the inscribed circle is equiangular to the triangle $I_1 I_2 I_3$.

Ex. 14. The orthocentre of an acute-angled triangle is the in-centre and the corners are the ex-centres of the pedal triangle.

Ex. 15. **I** is the in-centre and I_1, I_2, I_3 are the ex-centres of a triangle ; prove that the circumcircle of the triangle bisects II_1, II_2 and II_3.

THE CIRCUMCIRCLE

8. THEOREM. *If* **S** *is the circum-centre and* **O** *the orthocentre of a triangle* **ABC**, *prove that the* $\angle SAB =$ *the* $\angle OAC$, *the* $\angle SBA =$ *the* $\angle OBC$, *the* $\angle SCB =$ *the* $\angle OCA$.



Draw **SF** perp. to **AB**.

Then **SF** bisects the $\angle ASB$; so that the $\angle ASF = \frac{1}{2} \angle ASB =$ the $\angle C$.

Now the $\angle SAB$

= the compt. of the $\angle ASF$

= the compt. of the $\angle C$.

= the $\angle OAC$.

Similarly the other relations may be proved.

Ex. 16. If **S** is the circum-centre of a triangle **ABC**, prove that **SA, SB, SC** are perpendicular to the sides of the pedal triangle of **ABC**.

9. THEOREM. *If the altitudes of a triangle are produced to cut the circum-circle, then the part of each intercepted between the orthocentre and the circumcircle is bisected by the corresponding side of the triangle.*

Let the altitude **AP** be produced to cut the circum-circle at **D**.

To prove that **OD** is bisected by **BC** at **P**. [See Fig. of (8)].

Join **BD**.

Then the $\angle OBP =$ the $\angle OAQ$ or **DAC** = the $\angle DBC$.

Now in the Δs **BPO, BPD,**

∵ the $\angle OBP =$ the $\angle DBP$ (proved), the angles at **P** are rt. angles and **BP** is common to both ;

∴ **OP = DP.**

Similar proofs for the other altitudes.

Question. If the angle **A** be (i) a right angle and (ii) an obtuse angle, shew by diagrams what lines are bisected by **BC**.

Ex. 17. **AK** is the diameter of the circum-circle of the triangle **ABC** whose orthocentre is **O** ; prove that **BOCK** is a parallelogram, and **BC, OK** bisect each other.

The $\angle$s **ABK, ACK** [Fig. of (8)] in a semicircle are rt. angles ; ∴ **CR** is ∥ to **BK** and **BQ** to **KO**.

Ex. 18. The join of the orthocentre to the middle point of a side of a triangle and the diameter of the circum-circle drawn through the opposite vertex intersect at the same point on the circum-circle.

This follows at once from Ex. 17.

Ex. 19. If **S** is the circum-centre of a triangle **ABC** and **E, F, G** the middle points of **BC, AB** and **AC** respectively, shew that $SE=\frac{1}{2}OA$, $SF=\frac{1}{2}OC$, $SG=\frac{1}{2}OB$.

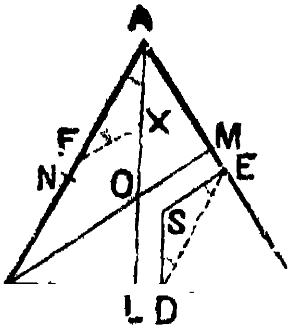
Draw the diameter **AK** [see Fig. of (8)]; then **OE** produced passes through **K** [Ex. 18.]

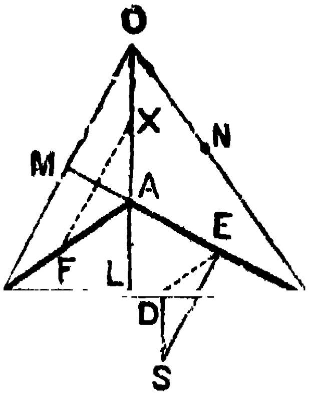
Ex. 20. The altitude of a triangle with respect to a side and the join of the orthocentre to the middle point of the side are produced to cut the circum-circle; prove that the join of the points of section is parallel to the side.

10. THEOREM. *The distance of the circumcentre from any side of a triangle is half the distance of the orthocentre from the opposite vertex.*

Let **ABC** be a triangle.

Let **L, M, N,** be the feet of the altitudes; so that **AL, BM,** are perp. to **BC** and **AC,** and their point of intersection **O** is the orthocentre.





Let **D, E, F,** be the middle pts. of **BC, CA** and **AB,** and **S** the circumcentre. Thus **SD. SE** are perps. on **BC** and **CA**; so that **SD** is the distance of **S** from **BC,** and **SE,** the distance of **S** from **CA.**

To prove that $SD=\frac{1}{2}OA$, $SE=\frac{1}{2}OB$, and $SF=\frac{1}{2}OC$.

Let **X** be the middle pt. of **OA**. Join **DE, FX**.

Now **DE** is parl. to and half of **AB**; ∴ $DE=AF$.

Also **SD,** is ∥ to **XA** and **DE** to **AB**; ∴ the $\angle SDE=$ the $\angle XAF$.

Again **SE, FX** are each parl. to **BM**; ∴ **SE** is ∥ to **FX**. And **DE** is ∥ to **AF** ∴ the $\angle DES=$ the $\angle AFX$.

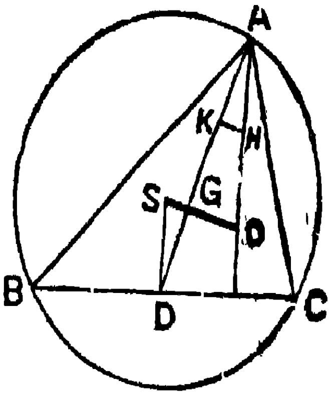
Hence the △s **SDE, XAF** are congruent; so that $SD=AX=\frac{1}{2}OA$, and $SE=XF=\frac{1}{2}OB$.

In the same way it may be proved that $SF=\frac{1}{2}OC$.

Obs. For another proof, see Ex. 19. A simpler proof however is by similar triangles **SDE, OAB.**

II. THEOREM. *The circumcentre, the orthocentre and the centroid of a triangle are collinear.*

Let **S** and **O** be the circumcentre and the orthocentre of the triangle **ABC**.



Join **OS** and draw **SD** perp. to **BC**.

Thus **BC** is bisected at **D**.

Join **AD** cutting **OS** at **G**.

To prove that **G** is the centroid.

Bisect **AO** at **H** and draw **HK** parl. to **OS**.

Now the ∠**AHK**=the ∠**AOG**=the alt. ∠**DSG** (∵ **SD** is ∥ to **AO**);
and the ∠**AKH**=the ∠**AGO**
=the vert. opp. ∠**DGS**.

Also **AH**=½**OA**=**SD**. [*App* III, (10)

∴ the △s **AHK**, **DSG** are congruent, so that **AK**=**DG**.

Again **HK** bisects **AG**; ∴ **AK**=**KG**=**GD**.

Hence **G** is the centroid of the △**ABC**.

Otherwise* Because **OA** is ∥ to **SD**;
∴ the ∠**AOG**=the alt. ∠**DSG**,
and the ∠**AGO**=the vert. opp. ∠**DGS**;
∴ the △s **AOG**, **DSG** are equiangular;
∴ **AG** : **DG**=**AO** : **DS**=2 : 1. [V. 3.

Thus **G** is the centroid of the △**ABC**.

Cor **G** *is a point of trisection of* **OS**.

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* This proof depends on Book V.

THE PEDAL LINE

12. THEOREM. *If from any point on the circum-circle of a triangle perpendiculars are drawn to the sides, the feet of the perpendiculars are collinear* ; and conversely, *if the feet of the perpendiculars from a point to the sides of a triangle are collinear, the point lies on the circum-circle of the triangle.*

Let **P** be a pt. on the circum-circle of the △**ABC** ; and let perps. **PL**, **PM**, **PN** be drawn to the sides.

To prove that the pts. **L**, **M**, **N**, are collinear.

Join **LM**, **MN**, **PA**, **PB**.

Now the angles at **M** and **N** are rt. ∠s, thus the quadl. **PMAN** is cyclic.

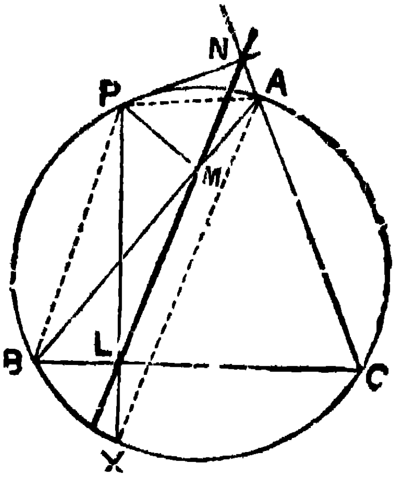
∴ the ∠**PMN** = the ∠**PAN**

= the ∠**PBC**, ∵ the quadl. **PBCA** is cyclic [III. 9, *Cor.*]

= the supt. of the ∠**PML** ;

that is, the ∠s **PMN**, **PML** are together equal to 2 rt. ∠s.

∴ **LM**, **MN** are in the same st. line.



Converse. Let **P** be a pt. from which perps. **PL**, **PM** and **PN** are drawn to the sides of the △**ABC** ; and let **LMN** be a st. line.

To prove that **P** lies on the circum-circle of the △**ABC**.

Join **PA**, **PB**.

Because **PBLM** is a cyclic quadl. ; ∴ the ∠**PBL** or **PBC** = the ∠**PMN** = the ∠**PAN** in the same segment = the supt. of the ∠**PAC** ; thus the ∠s **PBC**, **PAC** are supplementary ; ∴ **P**, **A**, **C**, **B** are concyclic ; *i. e.*, **P** is on the circle round the △**ABC**.

Cor. *The join of the feet of any two perpendiculars will be the pedal line.*

Def. The line **LMN** is called the **pedal line** or **Simson line** of the point **P** with respect to the triangle **ABC**.

13. THEOREM. *If the perpendicular from any point on the circumcircle of a triangle to any side cuts the circumcirle, then the join of the point of section to the opposite vertex is parallel to the pedal line of the point.*

[Let **PL** produced cut the circum-circle at **X** [see Fig. of (12)]. Join **AX**. Then **AX** is the parl to **LMN**. Supply proof.]

14. THEOREM. *The pedal line of a point* **P** *makes with the side* **BC** *opposite the* $\angle$ **A**, *an angle which is the complement of the angle subtended by the arc* **AP** *at the circumference.*

[See fig. of (12)].

Proof. The $\angle NLC =$ the $\angle BPM =$ the compt. of the $\angle ABP$ which is subtended by the arc **AP** at the circumference.

Similarly the angle which **LMN** makes with **AC** is the compt. of the angle subtended by the arc **BP** and with **AB** the compt. of the angle subtended by the arc **CP**.

15. THEOREM. *The angle between the pedal lines of any two points* **P** *and* **Q** *on the circum-circle of a triangle is equal to the angle subtended by the arc* **PQ** *at the circumference.*

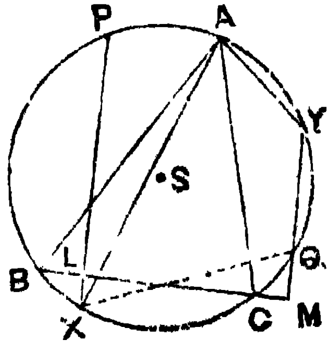
Let **ABC** be a triangle, whose circum-centre is **S**.

Draw **PL**, **QM**, perp. to **BC**, and produce them to meet the circum-circle in **X** and **Y**. Join **AX**, **AY**.

Then **AX**, **AY** are parl. to the pedal lines of **P** and **Q** respectively; and therefore the angle between the pedal lines of **P** and **Q** = the $\angle XAY$ by (13).

Join **QX**.

Thus the $\angle XAY =$ the $\angle XQM =$ the alt. $\angle PXQ$ which the arc **PQ** subtends at **X**.



Otherwise: Let α and β be the angles the pedal lines of **P** and **Q** make with **BC**, and θ the angle between them.

Then by (14), $\alpha = 90° -$ the $\angle AXP$, and $\beta = 90° -$ the $\angle AXQ$;

$\therefore\ \alpha + \beta = 180° -$ (the $\angle AXP +$ the $\angle AXQ) = 180° -$ the $\angle PXQ$.

But $\alpha + \beta$ is the suppt. of θ;

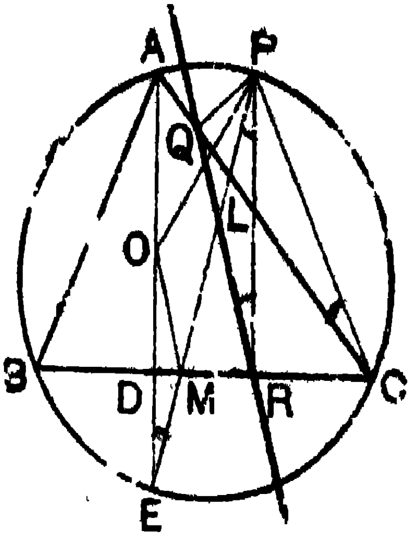
$\therefore\ \theta =$ the $\angle PXQ$.

Ex. 21. **PQ** is any diameter of the circum-circle; shew that the Simson lines of **P** and **Q** are at right angles.

[This may follow as a corollary to *App.* III, (15).]

Ex. 22. **PQ** is any diameter of the circum-circle; shew that the perpendiculars drawn from **P** and **Q** to their pedal lines respectively intersect in a point on the circle.

16. THEOREM. *The straight line joining the orthocentre of a triangle to any point on the circumcircle is bisected by the pedal line of that point.*



Let **O** be the orthocentre of the Δ**ABC** and **QR** the pedal line of **P**; **P** being any pt. on the circum-circle.

To prove that **OP** is bisected by **QR**.

Produce **AO** to cut **BC** in **D** and the circum-circle in **E**. Join **PE** cutting **QR** in **L** and **BC** in **M**. Join **PC, OM**.

Then because **PQRC** is a cyclic quadl.,

$\therefore$ the $\angle$**PRQ** = the $\angle$**PCQ**

= the $\angle$**PEA** in the same segment

= the alt. $\angle$**EPR**;

$\therefore$ **PL** = **LR**.

Hence, since the $\angle$**PRM** is a rt. angle, the $\angle$**LRM** = the $\angle$**LMR**;

$\therefore$ **ML** = **LR**; that is, **L** is the middle point of **PM**.

Again because **BC** bisects **OE** [by (9)] and the angles at **D** are rt. angles;

$\therefore$ the Δs **ODM, EDM,** are congruent;

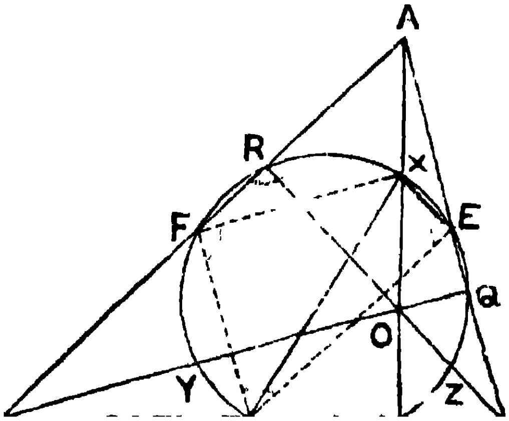
$\therefore$ the $\angle$**OMD** = the $\angle$ **EMD** = the vert. opp. $\angle$**LMR** = the $\angle$**LRM**;

$\therefore$ **QR** is parallel to **OM**.

Thus **RQ** being parallel to **OM** through **L**, the middle point of **PM**, bisects **OP**.

THE NINE-POINT CIRCLE

17. THEOREM. *In any triangle, the middle points of the sides, the feet of the perpendiculars from the vertices to the opposite sides, and the middle points of the lines joining the orthocentre to the vertices lie on a circle.*



In the △**ABC**, let **D, E, F** be the middle points of **BC, CA, AB** ; **P, Q, R** the feet of the perps. to these sides ; **X, Y, Z** the middle point of **OA, OB, OC, O** being the orthocentre.

To prove that the nine pts. **D, E, F** ; **P, Q, R** ; **X, Y, Z** lie on a circle.

Join **FX, FD, EX, ED, DX**.

Now **F** and **X** are the middle pts. of **AB** and **AO**, ∴ **FX** is ‖ to **BO** or **BQ** ;

and **F** and **D** are the middle pts. of **AB** and **BC**, ∴ **FD** is ‖ to **AC**;

∴ the ∠**DFX** = the angle bet. **BQ** and **AC** = a rt. angle.

Similarly it may be proved that the ∠**DEX** = a rt. angle.

Also the ∠**DPX** is a rt. angle.

Hence the circle described on **DX** as diameter, passes through **D, E, F, X** and **P** ; that is, **P** and **X** lie on the circle which passes through **D, E** and **F**.

Similarly it may be shewn that **Q, Y** and **R, Z** lie on the same circle.

Thus the nine pts. **D,E,F** ; **P,Q, R** ; **X,Y, Z** are concyclic.

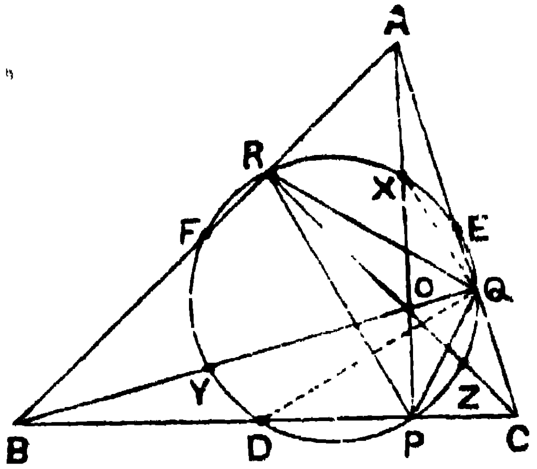
Cor. **DX, EY** and **FZ** *are diameters of the* N. P. *circle.*

Note. Since the circle passes through the nine points, it is called the **nine-point circle** of the triangle **ABC**.

ALTERNATIVE PROOF.

The letters have the same meaning as before.

Join **PQ, QR, RP**, and circumscribe a ⊙ about the △**PQR**. Join **QX**. **AQOR** is a cyclic quadl. ∴ the ∠**OAQ**=the ∠**ORQ**.



In the rt. ∠d △ **AQO**, **X** is the middle pt. of the hypot. **AO** ;
∴ the ∠**OXQ**=2∠**OAQ**.

Also **CR** bisects the ∠**PRQ** of the pedal △**PQR** ;
∴ the ∠**PRQ**=2∠**ORQ**.
∴ the ∠**PRQ**=the ∠**OXQ** or **PXQ**.
∴ **X** is on the circle **PQR**.

Similarly it may be shewn that **Y** and **Z** are on ⊙ **PQR**.

Again join **QD**.

From the rt. ∠d △ **BQC**, the ∠ **CDQ**=2∠**PBQ**.

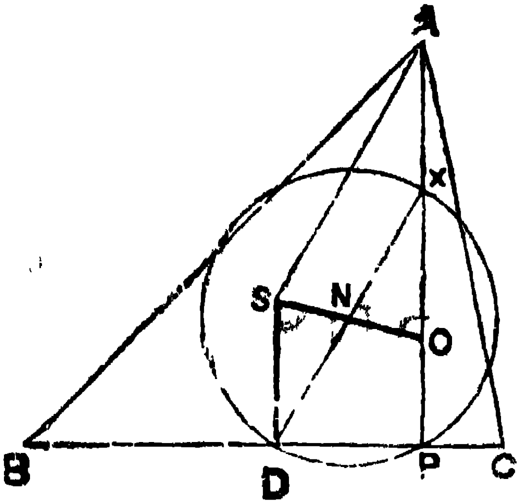
And the ∠**PBQ** or **PBO**=the ∠**PRO**, ∵ the quadl. **PBRO** is cyclic;
∴ as before the ∠**CDQ** or **PDQ**=the ∠**PRQ**.
∴ **D** is on the circle **PQR**.

Similarly **E** and **F** may be shewn to be on the ⊙ **PRQ**.

Hence the nine pts. **D,E,F** ; **P,Q,R,** ; **X,Y,Z** are concyclic.

We may also enunciate the Theorem otherwise thus :— *The circumcircle of the pedal triangle is the nine-point circle of the original triangle.*

18. THEOREM. *The centre of the nine-point circle is the middle point of the line joining the orthocentre to the circum-centre ; and its radius is half the radius of the circumcircle.*



Let **O** be the orthocentre, **S** the circum-centre and **DPX** the N. P. ⊙ of the △**ABC** ; so that **D** and **X** are the middle pts. of **BC** and **OA**.

To prove that (i) the *middle* pt. of **OS** is the *centre* of the N. P. circle ; (ii) the *radius* of the N. P. circle=*half the radius* of the circum-circle.

Join **SD** ; then **SD** is perp. to **BC** and therefore parl. to **AP**.

Join **DX**, intersecting **OS** at **N**.

(i) Now **SD**=**OX** by (10) ; and it is ∥ to **AP**;

∴ the ∠**NSD**=the ∠**NOX** ; also the ∠s at **N** are equal ;

∴ the △s **NSD, NOX** ; are congruent ;

∴ **ON**=**NS** and **NX**=**ND**.

Hence **N** is the middle pt. of both **OS** and **DX**.

But **DX** is a diam. of the N. P. circle ;

∴ **N** is the centre of the N. P. circle, and it is the middle pt. of **OS**.

(ii) Again **NX** is a radius of the N. P. ⊙, and it joins the middle pts. of **OS** and **OA** ;

∴ **NX** = ½ **SA** = half the radius of the circum-circle.

Note. We have seen that **O, G** and **S** are collinear [see (11)] and **N** is on **OS**.

Hence *the* **circum-centre,** *the* **orthocentre,** *the* **centroid** and *the* **nine-point centre** *are collinear.*

Ex. 23. Shew that in a right-angled triangle the nine-point centre is the middle point of a median.

Ex. 24. The nine-point circle of the triangle **ABC** is also the nine-point circle of each of the triangles **AOB, BOC, COA.**

[Note that all these triangles have the *same* pedal triangle]

Ex. 25. The circum-circles of the triangles **ABC, BOC, COA** and **AOB** are all equal. [See Ex. 9]

Ex. 26. The locus of the middle point of any line drawn from the orthocentre to the circumference of the circum-circle is the nine-point-circle.

Ex. 27. The intersection of the pedal line of any point and its join to the orthocentre lies on the nine-point circle.

Ex. 28. The Simson lines of the extremities of any diameter of the circum-circle of a triangle intersect at right angles on the nine-point circle.

MISCELLANEOUS EXERCISES.

1. **AB, AC** are any two chords of a circle, and **D, E** the mid, points of the arcs **AB, AC** ; prove that **DE** cuts off equal portions from **AB** and **AC.**

2. The internal bisector of the vertical angle **A** of a triangle **ABC** meets the circumcircle at **D** ; **DE, DF** are perpendicular to **AB** and **AC** respectively ; prove that $AE = \frac{1}{2}(AB + AC)$.

3. Two equal circles intersect at **A** and **B,** so that the centre of each is on the circumference of the other ; **CAD** is any straight line drawn through **A,** cutting the circles at **C, D** ; prove that **BCD** is always an equilateral triangle.

4. The tangent from any point on the outer of two concentric circles to the inner circle is constant

5. **ABCD** is a square ; circles are described with centres **B** and **C,** and radius equal to a side of the square, intersecting at **E** within the square ; prove that the $\angle BDE = \frac{1}{3}$ of a right angle.

6. Two triangles **ABC, XYZ** are inscribed in the same circle, the points being in order **A, X, B, Y, C, Z** ; so that arc **AX** = arc **BY** = arc **CZ.** Prove that the triangles are congruent.

7. If two circles intersect, their common tangents subtend supplementary angles at either point of intersection.

8. Two radii of a circle are perpendicular to each other and a tangent cuts them when produced ; prove that the tangents to the circle drawn from the points of section are parallel.

9. Four circular coins of different sizes are placed upon a table, so that each one touches two, and only two, of the remaining three ; shew that the four points of contact are concyclic.

10. **AB** is a chord of a circle and **C** a given point in **AB** ; find a point in the circumference at which both **AC, CB** subtend equal angles. How many such points are there ?

11. **P, Q** are any points on arcs of two segments described on a given straight line **AB** and on the same side of it ; ∠s **PAQ, PBQ** are bisected by **AR, BR** intersecting at **R** ; shew that ∠**ARB** is constant.

12. Circles are described on the sides of a quadrilateral as diameters ; prove that the common chord of any two adjacent circles is parallel to the common chord of the other two.

13. Equilateral triangles are described externally on the sides of any triangle ; prove that the lines joining the vertices of the triangle to the opposite vertices of the equilateral triangles are concurrent.

Hence shew that the circum-circles of the equilateral triangles intersect at the point of concurrence.

14. Two equal circles touch each other externally and through the point of contact chords are drawn, one to each circle, perpendicular to one another ; prove that the join of the other extremities of these chords is equal and parallel to the join of the centres.

15. The point of the intersection of the diagonals of a rhombus is the centre of the circle that can be inscribed in it.

16. **AB, CD** are diameters of a circle of centre **O** ; prove that the tangents at **A, B, C** and **D** form a quadrilateral whose diagonals intersect at **O**.

17. Construct a rhombus having given its angles and the radius of its inscribed circle.

18. From the ends of a diameter of a given circle, perpendiculars are drawn to a chord of fixed length ; prove that the sum or difference of these perpendiculars is constant for all positions of the diameter and the chord.

19. **ACB** is a triangle right-angled at **C**, and **O** is the point of intersection of the diagonals of the square described on **AB** ; shew that **OC** bisect the ∠**ACB** or its supplement.

20. The opposite sides of a cyclic quadrilateral are produced to meet at **P** ; prove that the perpendicular drawn to the bisector of the angle **P** from the intersection of the diagonals bisects the angle between them.

21. **P** and **Q** are the points of contact of tangents **OP, OQ** drawn to a circle whose centre is **C** ; **PC, OQ** are produced to meet at **L**, and **QC, OP** at **M** Prove that **LM** is parallel to **PQ**.

22. **AB, CD** are parallel diameters of two circles ; if their ends are joined directly or transversely by a line cutting the circles at two points, prove that the tangents at these two points are parallel.

23. **AB, AC** are chords of a circle ; **BD** is drawn parallel to the tangents at **A** cutting **AC** in **D** ; shew that **AB** in a tangent to the circum-circle of the triangle **BCD**.

24. **AB** is a diameter of circle and **PQ** is a constant arc ; prove that **AP**, **BQ** intersect at a constant angle.

If **AP**, **BQ** intersect at **O**, find the locus of **O**.

25. **P** is a point on the circum-circle of a triangle **ABC**, and the sides of a triangle **DEF** are parallel to **PA**, **PB** and **PC** ; prove that the triangles **DEF** and **ABC** are equiangular.

26. Two circles cut at **A** and **B** ; a chord **CAD** is drawn terminated by th [illegible]
E ; prove that **B**, **C**, **D**, **E** are concyclic.

27. **If two chords of a circle cut at right angles within or without the circle the sum of the squares on their segments is equal to the squares on the diameter.**

[Let **AC**, **BD** cut at rt. ∠s at **O** within the ⊙ [see Fig. *App.* III, (5)]. Draw diam. **AP** ; join **AB**, **BP**, **PC**, **CD**. Since **PC** is ∥ to **BD**, ∴ arc **CD**=arc **BP**, ∴ **CD**=**BP**.

$$\text{Now } OA^2+OC^2+OB^2+OD^2=(OA^2+OB^2)+(OC^2+OD^2)$$
$$=AB^2+CD^2=AB^2+BP^2=AP^2.$$

The proof is similar when **AC**, **BD** intersect without the ⊙]

28. If the diagonals of a cyclic quadrilateral are at right angles, the sum of the squares on a pair of opposite sides is equal to the square on a diameter of the circle.

29. **AB** touches a circle at **A** ; **PQ** is a parallel to **AB** cutting the circle at **P**, **Q** ; prove that **PQ**, and the tangent at **P** or **Q** are equally distant from **A**.

30. **The internal bisector of the vertical angle of a triangle and the perpendicular bisector of the base intersect on the circum-circle.**

31. **AB** is a diameter of a circle ; **AC**, **AD** are any two chords cutting the tangent at **B** in **P** and **Q** ; prove that the ∠**PCQ**=the ∠**PDQ**.

32. **AH** is a tangent at **A** to a circle whose centre is **O** ; **HKL** is any chord ; **KQ** is drawn perpendicular to **HK**, meeting the circle again in **Q** and **OA** in **R** ; shew that **AQ** is parallel to **HR**.

33. **ABCD** is cyclic quadrilateral of which the diagonal **AC** bisects the ∠**BCD**. If **CE** bisects the exterior angle of the quadrilateral at **C**, meeting the circle at **E**, prove that **AE** bisects **BD**.

34. The direct common tangents to two circles which touch externally also touch the circle described on the join of the centres as diameter ?

35. If the diagonals of a quadrilateral are at right angles, then the feet of the perpendiculars drawn to the sides from the intersection of the diagonals are concyclic.

36. A circle is drawn about the △**ABC** ; the tangent at **C** cuts **AB** produced in **D** ; the circle with centre **D** and radius **DC** cuts **AB** in **E**. Shew that **EC** bisects the ∠**ACB**.

37. Given the radii and the common chord of two circles, describe the circles.

38. **AP** is a diameter of a circle circumscribing an equilateral triangle **ABC**; if **O** is the centre of the circle, shew that **OCP** is an equilateral triangle.

39. **ALMB** is a semicircle; **AL**, **BM** intersect in **K** and **AM**, **BL** in **O**; **KO** is produced to meet **AB** in **N**. Prove that if **L** and **M** vary so as to keep the angle **LNM** constant, then the line **LM** is constant.

40. If on the sides of any triangle equilateral triangles are constructed externally, the joins of the circum-centres of these three triangles form another equilateral triangle. [See Ex. 13.]

41. Two equal circles intersect at **A** and **B**; if a third circle of centre **C** touches them, shew that **A**, **B**, **C** are collinear.

42. Circles are described on the sides of a right-angled triangle as diameters; prove that they are touched by a circle having for its centre the middle point of the hypotenuse and for its diameter the sum of the two sides.

43. The sum of the diameters to the incircle and the circumcircle of a right-angled triangle is equal to the sum of the sides containing the right angle.

44. Triangle formed by the chord of a circle, the tangent at one of its extremities and any line perpendicular to the diameter through its other extremity, will be isosceles.

45. **If three circles touch one another, the three common tangents at the points of contact are either concurrent or parallel.**

46. If three circles touch one another externally, the three common tangents at the points of contact are concurrent at the incentre of the triangle formed by joining the centres of the given circles.

47. A circle circumscribes a triangle. If each of the segments outside the triangle is turned about the side which is its chord until it is again in the plane of the paper, prove that the arcs of these segments intersect at the orthocentre of the triangle.

48. Construct a triangle having given the line along which one of the sides lies, and the feet of the altitudes from the extremities of that side.

49. If the middle points of the adjacent sides of a cyclic quadrilateral are joined, then the circles described about the four triangles thus formed are equal and touch the circle that circumscribes the quadrilateral.

[*Hint.* The diam. of each ⊙ is a radius of the ⊙ circumscribing the quadl.]

50. **AB** is a fixed chord of a circle and **XY** any chord passing through a fixed point **O** in **AB**; if **C** and **M** are the middle points of **AB** and **XY** respectively, shew that the ∠**CMX** is constant.

[*Hint.* **C**, **O**, **M** and the centre of the ⊙ are concyclic.]

51. **D, E, F** are the points of contact of the in-circle of the △**ABC**; if the feet of the altitudes of the △**DEF** are joined, prove that these joints are parallel to the sides of the △**ABC**.

52. If the altitudes of a triangle are produced to cut the circum-circle in **P, Q, R,** prove that the sides of the pedal triangle are parallel to the sides of the △**PQR**.

53. **A** and **B** are the centres of two circles which touch internally at **P** and also touch a third circle of centre **C** externally and internally respectively at **Q** and **R**; shew that the ∠**ACB** = 2∠**QPR**.

54. **AB** is a diameter of a semi-circle; **P, Q** are points on its arc; **AP, BQ,** intersect in **X** and **AQ, BP** in **Y**; prove that the tangents at **P** and **Q** intersect at the middle point of **XY**.

55. Two circles intersect at **A** and **B**, the centre **C** of one being on the circumference of the other; from a point **D** on the circumference of this circle, **DPB** and **DAQ** are drawn to cut the circle of centre **C** in **P** and **Q**; shew that **AP** is parallel to **BQ**.

56 Draw a circle to touch a given circle and a given straight line at a given point.

57. Draw a circle to touch a given straight line and a given circle at a given point.

58. Construct a triangle having given the feet of the altitudes.

59. Having given the base and the vertical angle of a triangle, shew that the triangle will be greatest when it is isosceles.

60. Two triangles **ABC, DEF** are inscribed in the same circle, so that **AD, BE, CF** intersect at a point **O**; prove that if **O** be the incentre of one of the triangles, it is the orthocentre of the other.

61. **ABC** is a triangle whose orthocentre is **O**; a circle through **B, O** and **C** cuts in **X** the median from **A** produced; prove that **AX** is twice this median.

62. If the altitude of a triangle **ABC**, whose orthocentre is **O**, be produced to cut the circumcircle in **P**, **Q** and **R**; prove that the vertices of the triangle are the centres of the circumcircles of the triangles **POQ, QOR** and **ROP**.

63 **If the in-centre and the ex-centres of a triangle are joined, the six joins are bisected by the circumcircle of the triangle.**

[*Hint.* The given △ is the pedal △ of the △ formed by joining the ex-centres.]

64. Construct a triangle having given the centres of its inscribed circle and of two of its escribed circles.

65. **A** is a point within a circle; **BC** is the chord bisected at **A**; **TP, TQ** are tangents whose chord of contact **PQ** passes through **A**; if **BC** produced cuts **TP, TQ** in **M** and **N**, prove that **AM** = **AN**.

[*Hint.* Join centre **O** to **A, P, Q, M, N** and prove ∠**OMN** = ∠**ONM**.]

66 If three equal circles have a common point of intersection, prove that the straight line joining two points of intersection, will be perpendicular to the straight line joining the other two points of intersection.

67. **AB** is a given straight line in which **C** is any point ; semi-circles are described on **AB**, **AC** and **BC** all on the same side of **AB** ; the perpendicular at **C** cuts the semi-circle on **AB** at **D** ; **DA**, **DB** are joined, cutting the smaller semi-circles at **P** and **Q** ; prove that **PQ** is a common tangent to the smaller semi-circles.

68. Through a point of intersection of two circles, draw a straight line terminated by the circumferences, so as to be bisected at that point

69. Prove that if the tangents at **B** and **C** to the circle **ABC** meet in **O**, the chord of the circle drawn through **O** parallel to **AB** will be bisected by **AC**.

[*Hint.* Let **K** be the centre. Prove that **D, K, O, C** are concyclic]

70. **ABC** is a triangle whose orthocentre is **O** ; **AO** cuts the circum-circle in a, **BO** in b and **CO** in c ; if any line drawn through **O** cuts **BC** in α, **CA** in β and **AB** in γ, prove that $a\alpha$, $b\beta$, $c\gamma$ meet on the circum-circle.

71. A number of circles touch a fixed line at a fixed point ; prove that the tangents at the points where the circles are cut by a fixed parallel to the given line all touch a fixed circle.

72. **ABCD** is a cyclic quadrilateral prove that the circles described on **AB**, **BC**, **CD** and **DA** as chords intersect again at four points which are concyclic.

73. The join of the incentre and the circumcentre of a triangle subtends at a vertex an angle equal to half the difference of the other two angles.

74. Construct a triangle having given the base, the vertical angle and the in-radius.

75. Construct a triangle having given the orthocentre, the nine-point circle and the middle point of one of the sides

76. If **O** is the orthocentre of a triangle **ABC**, prove that the triangle formed by the circumcentres of the triangles **AOB**, **BOC** and **COA** is congruent with the triangle **ABC**.

BOOK IV

RECTANGLES

BOOK IV

ON RECTANGLES

DEFINITIONS

1. A rectangle is a parallelogram one of whose angles is a right angle.

It has been already proved that all the angles of a rectangle are right angles. [See page 78, I. 24, cor. 1.]

Also it must be remembered that

the area of a rectangle = length × breadth ;
that is, the product of two adjacent sides.

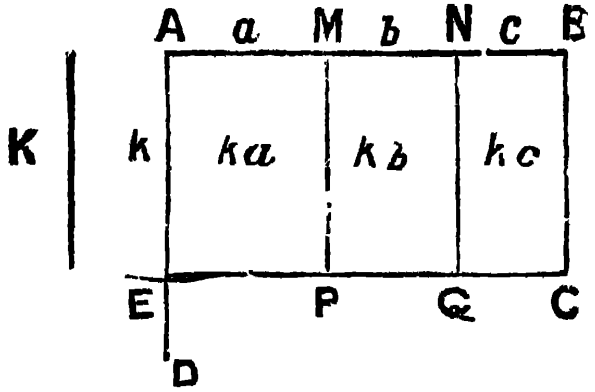
The expression "*product of two sides*" means "product of the *number of units* in one side by the *number of units* in the other."

2. If AB, AC are adjacent sides of a rectangle ABCD, then the rectangle ABCD is said to be contained by AB, AD and is denoted by "rect. AB, AD" or by "AB.AD."

So also a square whose side is AB is denoted by "sq. on AB" or by "AB^2."

THEOREM 1. [Euc. II. 1.]

If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line



Let AB and K be two straight lines; let AB be divided at M, N into any number of parts AM, MN, NB, containing a, b and c units of length; so that AB $=(a+b+c)$ units of length. Let the line K $=k$ units of length.

To prove that K.AB = K.AM + K.MN + K.NB;

that is, $k(a+b+c)=ka+kb+kc$.

Construction. Draw AD perp. to AB; and make AE equal to K. Through E, draw EC parl. to AB; and through M, N and B, draw MP, NQ and BC parl. to AE.

Proof. Then by construction the figs. AC, AP, MQ and NC are rectangles and AE = MP = NQ = BC = k units.

Now fig. AC = fig. AP + fig. MQ + fig. NC;

and fig. AC = rect. K,AB $=k(a+b+c)$ units of area;

fig. AP = rect. K,AM $=ka$..............................;

fig. MQ = rect. K,MN $=kb$..............................;

fig. NC = rect. K,NB $=kc$..............................;

∴ K.AB = K.AM + K.MN + K.NB;

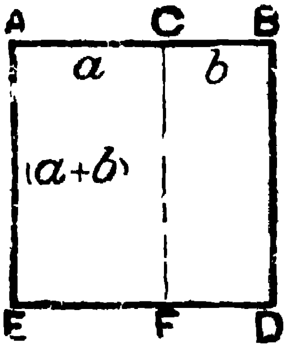
that is, $k(a+b+c)=ka+kb+kc$. Q.E.D.

Obs. **The geometrical theorem here illustrated and explained corresponds to the algebraical identity**

$$k(a+b+c+\ldots\ldots)=ka+kb+kc\cdots\ldots$$

THEOREM 2. [Euc. II. 2.]

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.



Let the st. line AB be divided at C into any two parts AC and CB, containing a and b units of length ; so that AB $=(a+b)$ units of length.

To prove that $AB^2=AB.AC+AB.CB$;

that is, $(a+b)^2=a(a+b)+b(a+b)$.

Construction. On AB, describe the square ABDE ; and draw CF parl. to AE or BD.

Proof. Then the figs. AF and CD are rectangles ; and

AB = AE = BD $=(a+b)$ units.

Now fig. AD = fig. AF + fig. CD ;

and fig. AD = the sq. on AB $=(a+b)^2$ units of area ;

fig. AF = rect. AE, AC $=a(a+b)$.................. ;

fig. CD = rect. BD, CB $=b(a+b)$................ ;

∴ $AB^2=AB.AC+AB.CB$;

that is, $(a+b)^2=a(a+b)+b(a+b)$.

Q. E. D.

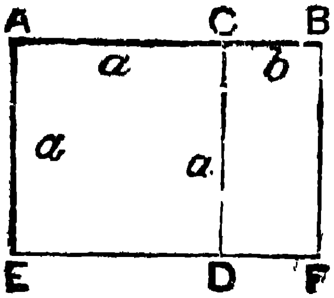
Obs. **The geometrical theorem here illustrated and explained corresponds to the algebraical identity**

$$(a+b)^2=a(a+b)+b(a+b).$$

Remark. This is a particular case of Theorem 1. Why ?

THEOREM 3. [EUC. II. 3.]

If a straight line is divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.



Let the st. line AB be divided at C into any two parts AC and CB, containing a and b units of length ; so that $AB = (a+b)$ units.

To prove that $AB.AC = AC^2 + AC.CB$;

that is, $$a(a+b) = a^2 + ab.$$

Construction. On AC, draw the square ACDE. Through B, draw BF parl. to CD, cutting ED produced at F.

Proof. Then the figs. AF and CF are rectangles ; and

$AC = AE = CD = a$ units.

Now fig. AF = fig. AD + fig. CF ;

and fig. AF = rect. AB, AE

= rect. AB, AC = $a(a+b)$ units of area ;

fig. AD = sq. on AC = a^2......... ;

fig. CF = rect. CD,CB = rect. AC,CB = ab.......... ;

$\therefore$ $AB.AC = AC^2 + AC.CB$;

that is, $$a(a+b) = a^2 + ab.$$ Q. E. D.

Obs. **The geometrical theorem, illustrated and explained here, corresponds to the algebraical identity**

$$a(a+b) = a^2 + ab.$$

Remark. This is a particular case of Theorem I. Why?

SEGMENTS OF A STRAIGHT LINE

Def. If a point is taken in a straight line, or in the line produced, the distances of the point from the extremities of the line are called its **segments.**

Thus if **P** is taken in **AB** (Fig. 1), or in **AB** produced (Fig. 2), then **AP** and **BP** are the segments of **AB**.

Fig. 1.

Fig. 2.

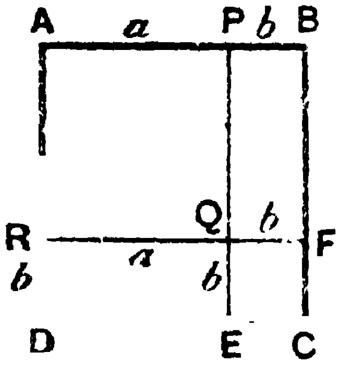
In Fig. 1, AB is divided **internally** at P, and P is the point of *internal division.*

In Fig. 2, AB is divided **externally** at P, P being the point of *external division.*

In internal division, AB is the **sum** of the segments; and in external division, AB is the **difference** of the segments.

THEOREM 4. [Euc. II. 4.]

If a straight line is divided *internally* at any point the square on the line is equal to the sum of the squares on the two segments *increased* by twice the rectangle contained by the segments.



Let the st. line AB be divided internally at P, so that AB is the *sum* of the two segments AP, PB.

Let $AP=a$ and $PB=b$ units of length, so that $AB=(a+b)$ units of length.

To prove that $AB^2=AP^2+PB^2+2AP.PB$,

that is, $(a+b)^2=a^2+b^2+2ab$.

Construction. On AB, AP, draw the squares ABCD and APQR; then AR falls along AD, and $PB=RD=b$ units.

Produce PQ to meet CD in E, and RQ to meet BC in F.

Proof. Then all the quadrilaterals in the figure are rectangles.

Because $QF=PB=RD=QE=b$ units;

∴ fig. QC is the sq. on PB.

Now fig. AC = fig. AQ + fig. QC + fig. DQ + fig. QB;

and fig. AC = sq. on $AB=(a+b)^2$ units of area;

fig. AQ = sq. on $AP=a^2$........................;

fig. QC = sq. on $PB=b^2$........................:

fig. DQ = rect. RQ, RD

= rect. AP, PB $=ab$........................:

fig. $QB = \text{rect. } PQ, PB$

$= \text{rect. } AP, PB = ab$ units of area ;

$\therefore AB^2 = AP^2 + PB^2 + 2\,AP.PB$;

that is, $$(a+b)^2 = a^2 + b^2 + 2ab.$$

Q.E.D.

Cor. *The square on any straight line is equal to four times the square on its half.*

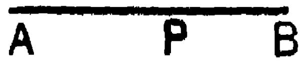
Obs. The geometrical theorem illustrated and explained here corresponds to the algebraical identity

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Hence the Theorem may be otherwise enunciated thus :—*The square on the sum of two straight lines is equal to the sum of the squares on the two lines together with twice the rectangle contained by the lines.*

The "two st. lines" are the *two segments* of AB.

Alternative Proof



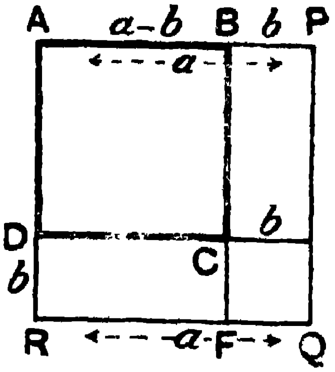
$AB^2 = AB.AP + AB.PB$ [IV. 2.

$= AP^2 + AP.PB + PB^2 + AP.PB$ [IV. 3.

$= AP^2 + PB^2 + 2AP.PB.$

THEOREM 5. [Euc. II. 7.]

If a straight line is divided *externally* at any point the square on the line is equal to the sum of the squares on the two segments *diminished* by twice the rectangle contained by the segments.



Let the st. line AB be divided externally at P; so that AB is the *difference* of the two segments AP, PB.

Let AP$=a$ and PB$=b$ units of length; so that AB$=(a-b)$ units of length.

To prove that $AB^2 = AP^2 + PB^2 - 2AP.PB$;

that is, $$(a-b)^2 = a^2 + b^2 - 2ab.$$

Construction. On AP, AB, draw the squares APQR and ABCD; then AD falls along AR, and PB$=$DR$=b$ units.

Produce DC to meet PQ in E, and BC to meet RQ in F.

Proof. Then all the quadrilaterals in the figure are rectangles. Because CE$=$PB$=$DR$=$CF$=b$ units;

∴ fig. CQ is the sq. on PB.

Now fig. AC = fig. AQ − fig. DQ − fig. EB
= fig. AQ + fig. QC − fig. DQ − fig. QB;

and fig. AC = sq. on AB $=(a-b)^2$ units of area;
fig. AQ = sq. on AP $=a^2$........................ ;
fig. QC = sq. on PB $=b^2$........................ ;
fig. DQ = rect. RQ,RD
= rect. AP,PB $=ab$........................ ;

fig. $QB = \text{rect. } PQ,PB$

$= \text{rect. } AP,PB = ab$ units of area ;

$\therefore\ AB^2 = AP^2 + PB^2 - 2AP.PB$;

that is, $(a-b)^2 = a^2 + b^2 - 2ab.$

Q.E.D.

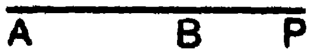
Obs. **The geometrical theorem illustrated and explained here corresponds to the algebraical identity**

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

Hence the Theorem may be otherwise enunciated thus :—*The square on the difference of two straight lines is equal to the sum of the squares on the two lines diminished by twice the rectangle contained by the lines.*

The "two st. lines" are the *two segments* of AB.

Alternative Proof



Because AP is divided internally at B,

$\therefore\ AB^2 + PB^2 + 2AB.BP = AP^2$; [IV. 4.

$\therefore\ AB^2 + 2PB^2 + 2AB.BP = AP^2 + PB^2$

But $PB^2 + AB.BP = AP.PB$; [IV. 3.

$\therefore\ AB^2 + 2AP.PB = AP^2 + PB^2$;

that is, $AB^2 = AP^2 + PB^2 - 2AP.PB.$

Note. The Theorems 4 and 5 may be included in one enunciation thus : *If a straight line is divided into two segments, the square on the line is equal to the sum of the squares on the two segments* **increased** *or* **diminished** *by twice the rectangle contained by the segments, according as the line is divided* **internally** *or* **externally**.

*THEOREM 6. [Euc. II. 9 and 10.]

If a straight line is bisected and also divided *internally* or *externally* into two unequal segments, the sum of the squares on the two segments is equal to twice the square on half the line together with twice the square on the line between the points of section.

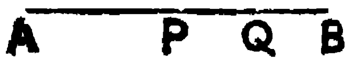


Fig. 1.

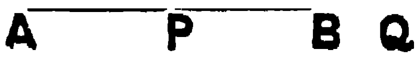


Fig. 2.

Let AB be the given st. line ; and let it be bisected at P and divided unequally at Q, internally in Fig. 1 and externally in Fig. 2.

To prove that $AQ^2+QB^2=2AP^2+2PQ^2$.

Proof. Because AQ is divided internally at P ;

$$\therefore AQ^2=AP^2+PQ^2+2AP.PQ \qquad [IV. 4.$$

And because QB is divided externally at P ;

$$\therefore QB^2=BP^2+PQ^2-2PB.PQ \qquad [IV. 5.$$

$$=AP^2+PQ^2-2AP.PQ \quad [\because BP=AP].$$

Hence $$AQ^2+QB^2=2AP^2+2PQ^2.$$

Q.E.D.

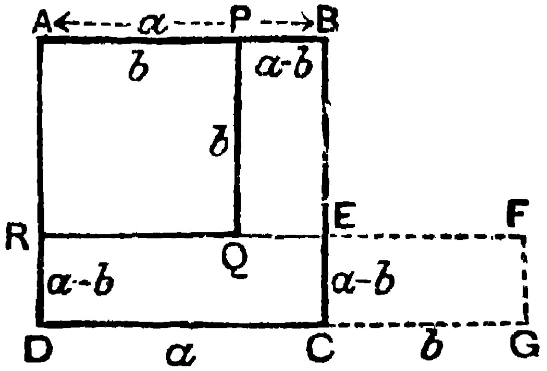
Note. Theorem 6 is thus an immediate deduction from Theorems 4 and 5.

It may also be enunciated thus : **The square on the sum of two straight lines together with the square on their difference is equal to twice the sum of the squares on the lines**

That is, $(AP+PQ)^2+(AP\sim PQ)^2=2AP^2+2PQ^2$

THEOREM 7. [Euc. II. 5 and 6.]

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.



Let AB, AP be two st. lines, containing a and b units of length ; and let AB be greater than AP. Let AB, AP be placed in the same st. line as in the figure.

To prove that $AB^2 - AP^2 = (AB + AP)(AB - AP)$;

that is, $a^2 - b^2 = (a + b)(a - b)$.

Construction. On AB, AP, draw the squares ABCD and APQR ;

then AR falls along AD, and RD = PB = AB − AP.

Produce RQ to meet BC at E, and DC to G, making CG = AP.

Draw GF parl. to BC meeting RE produced at F.

Proof. Then all the quadrilaterals in the figure are rectangles.

Because PQ = AP = CG, and PB = RD = CE ;

∴ rect. PE = rect. CF.

Now sq. AC − sq. AQ = rect. DE + rect. PE

= rect. DE + rect. CF = rect. RG.

Hence $AB^2 - AP^2 = (DC + CG).RD = (AB + AP)(AB - AP)$;

that is, $a^2 - b^2 = (a + b)(a - b)$.

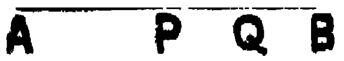
Q.E.D.

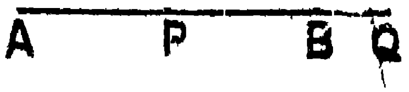
Obs. **The geometrical theorem illustrated and explained here corresponds to the algebraical identity**

$$a^2 - b^2 = (a + b)(a - b).$$

Cor. 1. *If a straight line is bisected and also divided internally or externally into two unequal segments, the rectangle contained by the two segments is equal to the difference of the squares on half the line and on the line between the points of section.*

[This is the Enunciation of Euclid II. 5 and 6.]





Let AB be bisected at P and divided unequally at Q, internally in Fig. 1 and externally in Fig. 2.

Then $AQ.QB = (AP + PQ)(PB - PQ)$ [Fig. 1.]

$\quad$ or $(AP + PQ)(PQ - PB)$ [Fig. 2.]

$\quad = (AP + PQ)(AP - PQ)$ or $(PQ + AP)(PQ - AP)$

$\quad = AP^2 - PQ^2$ or $PQ^2 - AP^2$.

Cor. 2. *If a straight line is bisected and also divided internally or externally into two unequal segments, the difference of the squares on the two segments is equal to twice the rectangle contained by the given line and the line between the points of section.*

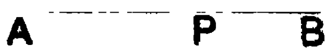
Let AB be bisected at P and divided unequally at Q, internally in Fig. 1 and externally in Fig. 2.

Then $AQ^2 - QB^2 = (AQ + QB)(AQ - QB)$

$\quad = AB.2PQ$ [Fig. 1] or $2PQ.AB$ [Fig. 2.]

$\quad = 2AB.PQ$.

Alternative Proof



Because AB is divided at P,

$\therefore AB^2 = AP^2 + PB^2 + 2AP.PB.$ [IV. 4.

$\therefore AB^2 - AP^2 = PB^2 + AP.PB + AP.PB$

$= AB.PB + AP.PB$ [IV. 3.

$= (AB + AP).PB$ [IV. 1.

$= (AB + AP)(AB - AP).$

EXERCISES LXIII.

Illustrate and explain by a geometrical figure the algebraical identity

1. $(2a)^2 = 4a^2.$

2. $a(b+c) = ab + ac.$

3. $a(b-c) = ab - ac.$

4. $(a+b)(c+d) = ac + bc + ad + bd.$

5. $(a-b)(c-d) = ac - bc - ad + bd.$

6. $(x+a)(x+b) = x^2 + ax + bx + ab.$

7 **Prove that the parallelograms about the diagonal of a square are also squares.**

8. Explain how figures 1, 2 and 3 respectively illustrate the algebraical identities (i), (ii) and (iii) given below.

(i) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$

(ii) $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2.$

(iii) $(a+b)^2 - (a-b)^2 = 4ab.$

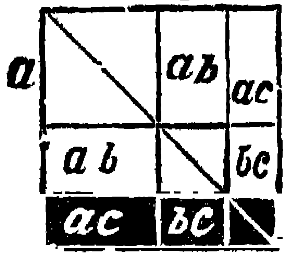


FIG. 1.

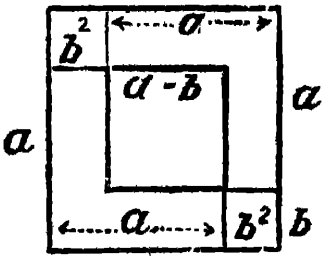


FIG. 2.

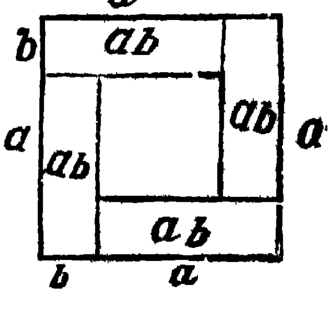


FIG. 3.

9. Shew that Identity (ii) in Ex. 8, corresponds to Theorem 6 [see the alternative Enunciation given in the **Note,** Theorem 6].

10. Shew that Identity (iii) in Ex. 8, corresponds to the Theorem which may be enunciated thus:—**The square on the sum of two straight lines diminished by the square on their difference is equal to four times the rectangle contained by the lines.** [Euc. II. 8.]

11. **Construct a rectangle equal to the difference of two given squares.** [Use II. 13, *Cor.* 1.]

12 **A, B, C, D** are four points taken in order on a straight line; shew that $\mathbf{AB.AC+BC.BD+AB.BC=AC^2=BC.CD}$.

13. **A, B, C, D** are four points in order on a straight line; prove that

$$\mathbf{AC^2+BD^2+2AB.CD=AD^2+BC^2}.$$

14. **AB** is divided at **D** and **C**, so that **AD=BC**; prove that

$$\mathbf{AB^2+CD^2=2AC^2+2BC^2}.$$

[*Hint.* **AB=AC+BC.** And **CD** is divided externally at **A**; use II. 5.]

15. **ABCD** is a rectangle and **AE** is drawn to cut **CD** in **E**; if **F** is the middle point of **CE**, shew that

$$\mathbf{AC^2-AE^2=4CF.DF}.$$

16. **AB** is divided at **C**, so that $\mathbf{AC^2=2BC^2}$; prove that the sum of **AB** and **BC** is equal to the diagonal of the square on **AB**.

[*Hint.* Produce **AB** to **D**, making **BD=BC**. Then **CD** is bisected at **B** and divided externally at **A**.]

PROJECTIONS

The **projection of a point** on a given straight line is the foot of the perpendicular drawn from the point to the straight line.

Thus p is the projection of **P** on **XY**; so also q of the point **Q**.

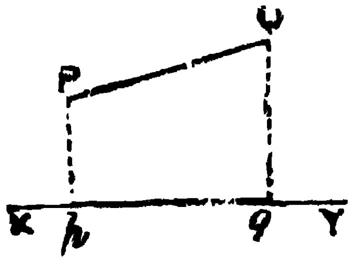


Fig. 1.

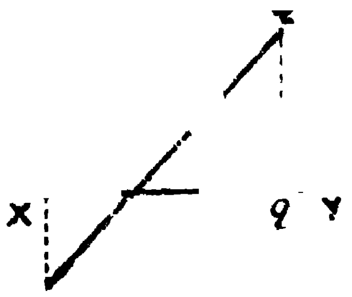


Fig. 2.

The **projection of a line PQ** on a given straight line **XY** is that portion of **XY** which is between the feet of the perpendiculars drawn from P and **Q** to **XY**; *i.e.*, the projection of **PQ** is the intercept between the projections of its extremities.

Thus pq is the projection of the line **PQ** on the st. line **XY**.

Since [Fig. 2] the projection of the pt. O on XY is the pt. O itself, the projection of OP on XY is Op and of OQ is Oq; and the projection of the whole line PQ on XY is pq.

Again in the obtuse-angled △ABC [see Fig. IV. 8.] where the ∠ACB is obtuse, CD is the *projection of* AC *on* BC. Here BC is *produced.*

If PQ, PQ′, PQ″............be the different positions of the line PQ it is evident that as the slope of PQ *increases*, the projection of PQ *diminishes* and *vice-versâ.*

Hence *the projections of two equal and parallel straight lines on the same straight line are equal.* (Draw a figure and prove this.)

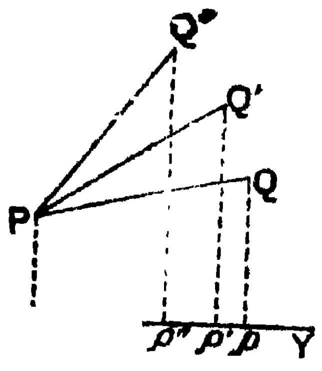


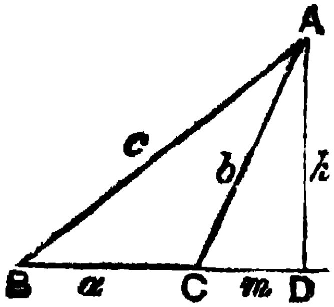
Fig. 3.

Question 1. When is the projection of a line on another line is equal to itself? When is it zero?

Question 2. Prove that if a straight line is bisected, its projection is also bisected.

THEOREM 8. [Euc. II. 12.]

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle *increased* by twice the rectangle contained by one of these sides and the projection on it of the other side.



Let ABC be a triangle, having the ∠ACB an obtuse angle.

Let AD be the perpendicular from A to BC produced; so that CD is the *projection* of the side AC on BC,

To prove that $AB^2 = BC^2 + AC^2 + 2BC.CD$.

Proof. Because the ∠D is a rt. angle;

$\therefore$ $AB^2 = AD^2 + BD^2$. [II. 5.

Now BD is divided internally at C, so that BD is the sum of BC and CD;

$\therefore$ $BD^2 = BC^2 + CD^2 + 2BC.CD$. [IV. 4.

Hence $AB^2 = AD^2 + BC^2 + CD^2 + 2BC.CD$.

But $AD^2 + CD^2 = AC^2$, $\because$ the ∠D is a rt. angle;

$\therefore$ $AB^2 = BC^2 + AC^2 + 2BC.CD$.

Q.E.D.

The following is an **algebraical proof.**

Let $AB = c$, $BC = a$, $AC = b$, $AD = h$ and $CD = m$ units; thus $BD = (a + m)$ units.

Then, since the ∠ADB is a rt. angle, $AB^2 = AD^2 + BD^2$.

That is, $c^2 = h^2 + (a+m)^2 = a^2 + (h^2 + m^2) + 2am$

$= a^2 + b^2 + 2am.$

THEOREM 9. [Euc. II. 13.]

In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing the acute angle *diminished* by twice the rectangle contained by one of these sides and the projection on it of the other side.

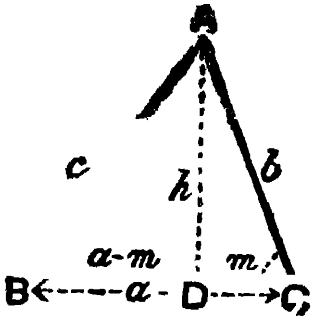


Fig. 1.

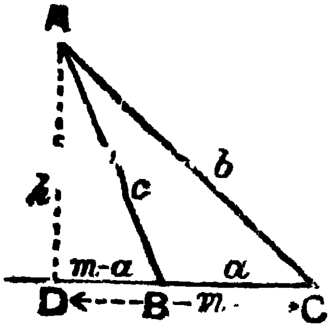


Fig 2.

Let ABC be any triangle, having the $\angle C$ an acute angle.

Let AD be perpendicular to BC, or BC produced; so that CD is the *projection* of the side AC on BC.

To prove that $AB^2 = BC^2 + AC^2 - 2BC.CD$.

Proof. Because the $\angle ADB$ is a rt. angle;

$\therefore AB^2 = AD^2 + BD^2$. [II. 5.

Now BD is divided externally at C, so that BD is the difference of BC and CD;

$\therefore BD^2 = BC^2 + CD^2 - 2BC.CD$. [IV. 5.

Hence $AB^2 = AD^2 + BC^2 + CD^2 - 2BC.CD$

But $AD^2 + CD^2 = AC^2$, $\because$ the $\angle ADB$ is a rt. angle;

$\therefore AB^2 = BC^2 + AC^2 - 2BC.CD$.

Q.E.D.

An **algebraical proof** is given below.

Let $AB = c$, $AC = b$, $BC = a$, $AD = h$ and $CD = m$ units.

Thus $BD = (a - m)$ units (Fig. 1),

or $BD = (m - a)$ units (Fig. 2).

Then since the $\angle ADB$ is a rt. $\angle$, $AB^2 = AD^2 + BD^2$.

That is, $c^2 = h^2 + (a-m)^2$ [Fig. 1],

or $c^2 = h^2 + (m-a)^2$ [Fig. 2].

Thus in both cases,

$$c^2 = a^2 + (m^2 + h^2) - 2am$$
$$= a^2 + b^2 - 2am.$$

Note

In a triangle **ABC**,

(i) $AB^2 = AC^2 + BC^2 + 2BC.CD$, when $\angle C$ is *obtuse* [IV. 8.];

(ii) $AB^2 = AC^2 + BC^2$, when $\angle C$ is a *right angle* [II. 5];

(iii) $AB^2 = AC^2 + BC^2 - 2BC.CD$, when $\angle C$ is *acute* [IV. 9.]

It is easy to see that in the second case, the perpendicular **AD** coincides with **AC** [see figure, IV. 8] and therefore the projection of **AC** on **BC** is 0. Thus $2BC.CD = 0$.

Hence the *three* Theorems 8, 9 and II. 5 may be included in *one* Enunciation thus :—*The square on a side of a triangle is* **greater than, equal to,** *or* **less than,** *the sum of the squares on the other two sides, according as the angle contained by these sides is* **obtuse, right** *or* **acute** ; *the difference in the cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.* (Syllabus).

EXERCISES LXIV.

[*Numerical and Graphical.*]

1. **ABC** is a triangle obtuse-angled at **C** ; if **AC**=5 cm., **BC** =6 cm. and the projection of **AC** on **BC**=2 cm. ; calculate the length of **AB**.

2. **ABC** is a triangle obtuse-angled at **C** ; **AB**=15 cm., **BC** =8 cm. and projection of **AB** on **BC**=11 cm. ; calculate the length of **AC**.

3. **ABC** is a triangle ; **AC**=8 cm. ; **BC**=12 cm. and the projection of **AC** on **BC**=3 cm. ; find **AB** (i) when the $\angle C$ is obtuse, and (ii) when it is acute.

4. **ABC** is a triangle ; **AC**=10 cm., **BC**=15 cm. and the projection of **AB** on **AC**=6 cm. ; find **AB**, if the $\angle C$ is acute.

5. The sides **AB, BC, CA** of triangle **ABC** are 6 cm., 5 cm., and 4 cm. respectively ; find the projections of **AB** on **BC**, of **BC** on **AC** and of **AC** on **AB**.

6. In a triangle **ABC**, the ∠**C** is obtuse ; if **AB** = 7 in., **BC** = 3 in. and **AC** = 5 in., calculate the length of the projection of **AB** on **BC**.

7. In a triangle **ABC**, **AB** = 2 in., **BC** = 5 in. and **AC** = 4 in. ; calculate the length of the projection of **AB** on **BC** and of **AC** on **AB**.

8. In Ex. 7, measure the altitude of the triangle **ABC** with respect to **AB** and **BC** respectively and hence find the area of the △**ABC**.

Do these two results agree ?

9. **The area of a triangle in terms of its sides is**

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, c denote the sides opposite to the angles **A**, **B**, **C** respectively and s the semi-perimeter, so that $2s = a + b + c$.

Let **ABC** be a triangle.

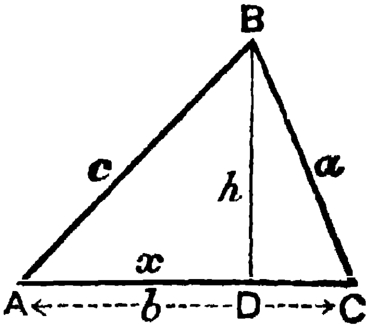
Draw **BD** perp. to **AC**.

Let **BD** $= h$ and **AD** $= x$.

Area of the △**ABC** $= \frac{1}{2}bh$ .. .. (i).

And $a^2 = b^2 + c^2 - 2bx$,

$\therefore x = \frac{b^2+c^2-a^2}{2b}$ ....(ii).



Also $h^2 = c^2 - x^2 = c^2 - \left(\frac{b^2+c^2-a^2}{2b}\right)^2$, from (ii)

$$= \frac{4b^2c^2 - (b^2+c^2-a^2)^2}{4b^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4b^2}$$

$$= \frac{\{(b+c)^2 - a^2\}\{a^2 - (b-c)^2\}}{4b^2}$$

$$= \frac{(b+c+a)(b+c-a)(a-b+c)(a+b-c)}{4b^2}.$$

Now $a+b+c = 2s$, $\therefore a+b+c-2a = 2s-2a$, or $b+c-a = 2(s-a)$.

Similarly $c+a-b = 2(s-b)$, $a+b-c = 2(s-c)$.

Hence

$$h^2 = \frac{2s.2(s-a).2(s-b).2(s-c)}{4b^2} = \frac{4s(s-a)(s-b)(s-c)}{b^2};$$

$$\therefore h = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)} \text{ .... (iii).}$$

Hence from (i) and (iii)

$$\text{Area reqd.} = \tfrac{1}{2}bh = \sqrt{s(s-a)(s-b)(s-c)}.$$

10. Find the area of a triangle whose sides are 5 ft., 6 ft. and 9 ft.

11. Find the area of a triangle whose sides are 6, 8 and 11 cm

12. Find the area of a triangle one side of which is 20″, the difference of the other two is 8″ and the sum of the three sides is 45″.

[*Theoretical.*]

13. The angle **ABC** of a triangle **ABC** is 120°; prove that

$$AB^2 = AC^2 + BC^2 + AC.CB.$$

14. The angle **ACB** of a triangle **ABC** is 60°; prove that

$$AB^2 = AC^2 + BC^2 - AC.CB.$$

15. **ABC is a triangle; perpendiculars BM, CN are drawn from any two angles B and C to the opposite side: if BM, CN intersect at O, prove that**

the rect. **OB,OM** = the rect. **OC,ON.**

Draw diagrams to illustrate the cases where **A** *or* **B** *or* **C** *is an obtuse angle.*

16. In a quadrilateral **ABCD**, the opposite angles at **B** and **D** are right angles; **AD, BC** are produced to meet at **E**; prove that
the rect. **AE,DE** = rect. **BE.CE**.

17. The angles **B** and **C** of a triangle **ABC** are acute; **BE, CF** are perpendiculars to **AC** and **AB**; prove that

the sq. on **BC** = the rect **AB,BF** + the rect. **AC,CE.**

What would this relation become if **B** be an obtuse angle?

18. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and the projection of the the base upon it.

19. The sum of the squares on the diagonals of a trapezium is equal to the sum of the squares on the two non-parallel sides together with twice the rectangle contained by the two parallel sides.

20. **ABC** is a triangle; on the side of **BC** remote from **A**, draw a square **BDEC**; shew that $AD^2 \sim AE^2 = AB^2 \sim AC^2$

21. The difference between the squares on the diagonals of a parallelogram is four times the rectangle contained by one of the sides and the projection upon it of the adjacent side.

*THEOREM 10.

In a triangle, the sum of the squares on any two sides is equal to twice the squares on half the third side together with twice the square on the median that bisects the third side. [**Apollonius' Theorem.**]

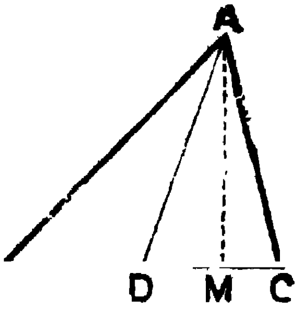


FIG. *1*.

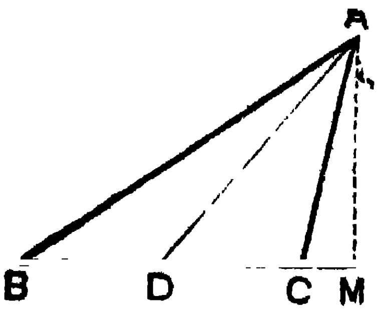


FIG 2.

Let ABC be a triangle, and AD the median that bisects BC.

To prove that $AB^2 + AC^2 = 2BD^2 + 2AD^2$.

Construction. Draw AM perp. to BC or BC produced; so that DM is the *projection* of AD on BC.

Proof. Suppose AB and AC to be unequal; then of the angles ADB, ADC, one is obtuse and the other acute.

Let the $\angle ADB$ be obtuse; consequently the $\angle ADC$ is acute.

Then in the $\triangle ADB$,

$\therefore AB^2 = BD^2 + AD^2 + 2BD.DM,$ [IV. 8.

And in the $\triangle ADC$,

$\therefore AC^2 = CD^2 + AD^2 - 2CD.DM$ [IV. 9.

$= BD^2 + AD^2 - 2BD.DM \; (\because CD = BD).$

Hence by addition,

$$AB^2 + AC^2 = 2BD^2 + 2AD^2.$$

Q.E.D.

Note. Suppose **AM** continually diminishes, so that the pt. **A** continually approaches nearer and nearer to the line **BC**. Then when **A** is indefinitely near to **BC**, we have by Apollonius' Theorem,

$$AB^2+AC^2=2BD^2+2AD^2\ ;$$

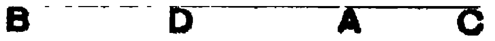


Fig. 1.

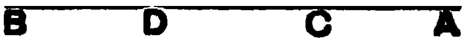


Fig. 2.

and this is always true. Hence ultimately when **A** coincides with **M**, *i.e.*, when **A** is a pt. on **BC** [Fig. 1], or on **BC** produced [Fig. 2], we get

$$AB^2+AC^2=2BD^2+2AD^2. \quad \text{[See Theorem 6.]}$$

EXERCISES LXV.

1. Calculate the base of a triangle whose sides are 4 and 5 cm. and whose median is 3·5 cm. Verify by drawing.

2. Calculate the medians of a triangle whose sides are 3, 4 and 6 cm. Verify by drawing.

3. **ABC** is a straight line, $AB=BC=2$ cm. ; **P** is a point outside **AB** ; if $PB=4$ cm., $PC=3$ cm., find **AP** and verify the result by measurement.

4. **ABC** is an isosceles triangle, having $AB=AC$; **BC** is produced to **D**, so that $CD=BC$; prove that $AD^2=AB^2+2BC^2$.

5. **ABC** is an isosceles triangle having $AB=AC$; **BD** is drawn bisecting **AC** ; prove that BC^2 is equal to twice the rectangle contained by the sum and difference of **AD** and **BD**.

6. The vertex **A** of an isosceles triangle **ABC** is joined to any point **D** in **BC** or **BC** produced ; **DA** is produced to **E**, so that **DE** is bisected at **A** ; shew that the difference of the squares on **BE, CE** is equal to the difference of the squares on **BD, CD**.

7. **ABC** is a triangle ; **D** is a point in **BC**, such that $AB^2+BD^2=AC^2+CD^2$; shew that the perpendicular from the middle point of **AD** to **BC** bisects **BC**.

8. **ABCD** is a rectangle and **P** any point ; prove that

$$PA^2+PC^2=PB^2+PD^2.$$

9. **The locus of a point which moves so that the sum of the squares on its distances from two fixed points is constant, is a circle.**

[If k = the constant, then

$$2BD^2+2AD^2=AB^2+AC^2=k \text{ [see figs. of IV. 10] };$$

$$\therefore AD^2=\tfrac{1}{2}k-BD^2,$$

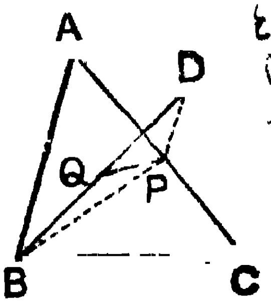
$$\text{or } AD=\sqrt{\tfrac{1}{2}k-BD^2}.$$

Thus **AD** is known, ∵ k and **BD** are known. Hence the required locus is a circle whose centre is the middle point of **BC** and radius **AD.**]

10. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

11. In any quadrilateral, the sum of the squares on the diagonals is equal to twice the sum of the squares on the joins of the middle points of the opposite sides.

12. **The sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals together with four times the square on the join of the middle points of the diagonals.**



Easy but Impt

[Let **ABCD** be a quadl. and let **P**, **Q** be the middle points of **AC**, **BD**.

To prove that

$$AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4PQ^2.$$

Join **PB**, **PD**.

From the Δs **ABC**, **ACD**, by Apoll. Theorem,

$$AB^2+BC^2=2AP^2+2BP^2,$$

$$\text{and } CD^2+DA^2=2AP^2+2DP^2.$$

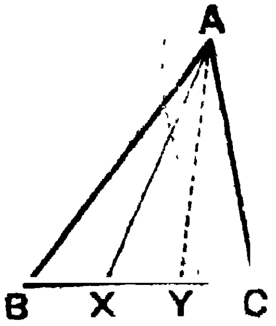
$$\begin{aligned}\therefore AB^2+BC^2+CD^2+DA^2&=4AP^2+2BP^2+2DP^2\\&=AC^2+2BP^2+2DP^2\\&=AC^2+4BQ^2+4PQ^2 \text{ from the } \Delta PBD\\&=AC^2+BD^2+4PQ^2.]\end{aligned}$$

13. If the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral is a parallelogram.

14. **In a triangle ABC, X is a point of trisection of the base BC adjacent to the angle B ; prove that**

$$2AB^2+AC^2=6BX^2+3AX^2.$$

[Let Y be the second point of trisection.



From the $\triangle ABY$,

$AB^2+AY^2=2BX^2+2AX^2$,

$\therefore\ 2AB^2+2AY^2=4BX^2+4AX^2$.

From the $\triangle AXC$,

$AX^2+AC^2=2AY^2+2XY^2$,

$\therefore\ 2AB^2+2AY^2+AX^2+AC^2$,

$=4BX^2+4AX^2+2AY^2+2XY^2$,

or $2AB^2+AC^2=6BX^2+3AX^2$, $\because$ $BX=XY$.]

15. Prove that the difference of the squares on the sides of a triangle is equal to three times the difference of the squares on the joins of the vertex to the points of the trisection of the base.

16. Prove that the sum of the squares on the sides of a triangle is equal to the sum of the squares on the joins of the vertex to the points of trisection of the base together with four times the squares on the line between the joins.

17. If G is the centroid of a triangle ABC, shew that

$$AB^2+AC^2=GB^2+GC^2+4GA^2.$$

18. A circle is described with the point of intersection of the diagonals of a parallelogram as centre ; prove that the sum of the squares on the joins of any point on its circumference to the angular points of the parallelogram is constant.

19. **Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.**

20. If G is the centroid of a triangle ABC, prove that

$$AB^2+BC^2+CA^2=3(GA^2+GB^2+GC^2).$$

21. C and D are two points in the diameter AB of a circle, equidistant from the centre ; if P is any point on the circumference, prove that

$$PC^2+PD^2=AC^2+AD^2. \quad \text{[Use IV. 6.]}$$

22 AB is a diameter of a semi-circle and CD a chord parallel to AB ; if P is any point in AB, prove that

$$PA^2+PB^2=PC^2+PD^2 \quad \text{[Use IV. 6].}$$

THEOREM 11. [Euc. III. 35.]

If two chords of a circle intersect either inside or outside the circle, rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

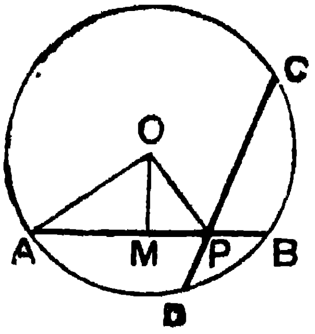


Fig. 1.

Let AB, CD be two chords of a circle ABC, whose centre is O and radius R.

(i) Let AB, CD intersect at a point P **inside** the circle.

To prove that *the rect.* AP,PB = *the rect.* CP,PD.

Construction. Draw OM perp. to AB.

Join OA, OP.

Proof. Because OM is perp. to the chord AB;

∴ OM bisects AB; so that AM = BM.

$$\begin{aligned}\text{Now } AP.PB &= (AM+MP)(BM-MP)\\ &=(AM+MP)(AM-MP)\\ &= AM^2-MP^2 && \text{[IV. 7.}\\ &=(AM^2+OM^2)-(MP^2+OM^2)\\ &=OA^2-OP^2 && \text{[II. 5.}\\ &=R^2-OP^2.\end{aligned}$$

Similarly it can be shewn that CP.PD = $R^2 - OP^2$.

Hence the rect. AP,PB = the rect. CP,PD.

Q.E.D.

(ii) Let AB, CD intersect at a point P **outside** the circle.

To prove that *the rect.* AP,PB = *the rect.* CP,PD.

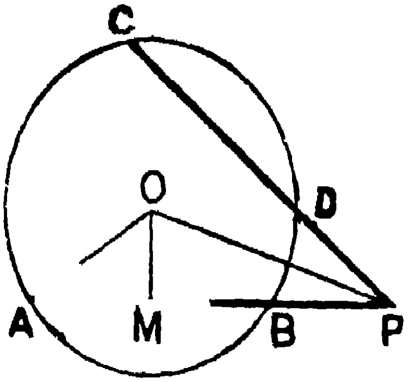


FIG. 2

Construction. Draw OM perp. to AB.
Join OA, OP.

Proof. Because OM is perp. to the chord AB ;
∴ OM bisects AB ; so that AM = BM.

$$\begin{aligned} \text{Now } AP.PB &= (MP+AM)(MP-BM) \\ &= (MP+AM)(MP-AM) \\ &= MP^2 - AM^2 & [IV.\ 7 \\ &= (MP^2+OM^2)-(AM^2+OM^2) \\ &= OP^2 - OA^2 & [II.\ 5. \\ &= OP^2 - R^2. \end{aligned}$$

Similarly it can be shewn that $CP.PD = OP^2 - R^2$.

Hence the rect. AP,PB = the rect. CP,PD.

Q.E.D.

Cor. 1. *If a chord of a circle passes through a point inside the circle, the rectangle contained by its segments is equal to the square on half the chord bisected at the point.*

In Fig. 1, if **AB** be a chord through **P** and **KL** a chord drawn so as to be bisected at **P**, then **OP** is perpendicular to **KL**.

Hence $KP^2 = OK^2 - OP^2 = R^2 - OP^2 = AP.PB.$

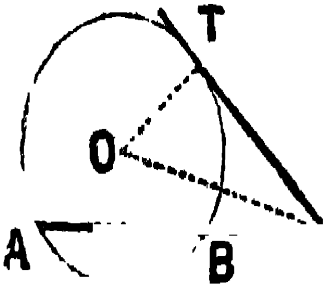
Cor. 2. *If a chord of a circle passes through a point outside the circle, the rectangle contained by its segments is equal to the square on the tangent from the point.* [Euc. III. 36]

For if **AB** be a chord through **P** and **PT** a tangent from **P**, then

$\because$ **PT** is perp. to the radius **OT**,

$\therefore PT^2 = OP^2 - OT^2 = OP^2 - R^2$

$= AP.PB.$



Note. Since $R^2 \sim OP^2$ is a *constant* quantity, we may enunciate Theorem 11 thus :—

If a chord of a circle passes through a fixed point either inside or outside the circle, the rectangle contained by its segments is constant.

Hence it follows that if **any number of chords of a circle pass through a fixed point either inside or outside the circle, the rectangles contained by their segments are all equal to one another.**

For each of them $= R^2 \sim OP^2$.

THEOREM 11. [ALTERNATIVE PROOF*.]

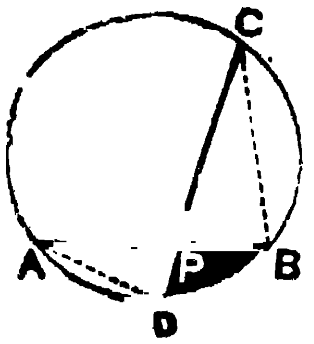


FIG. 1.

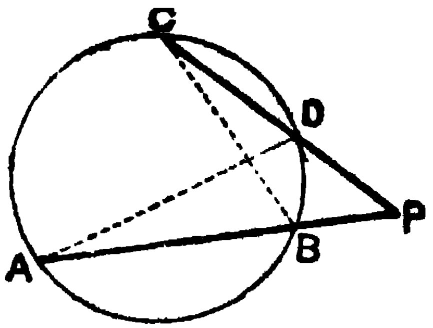


FIG. 2.

Let two chords AB, CD intersect at P, inside (Fig. 1) or outside (Fig. 2) the circle.

To prove that *the rect.* AP,PB = *the rect.* CP, PD.

Construction. Join AD, BC.

Proof. In the $\triangle^s$ APD, CPB,

the ∠APD = the ∠CPB, being vert. opp. angles in Fig. 1 and the same angle in Fig. 2,

and the ∠PAD = the ∠PCB, in the same segment ;

∴ the $\triangle^s$ APD, CPB are equiangular ;

∴ AP : CP = PD : PB ;

∴ AP.PB = CP.PD ;

that is, the rect. AP,PB = the rect. = CP,PD.

Q.E.D.

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* This proof should be taken up after the student has read Book V.

*THEOREM 12. [CONVERSE OF THEOREM 18.]

If two straight lines intersect, both internally or both externally, so that the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other, then the extremities of the lines are concyclic.

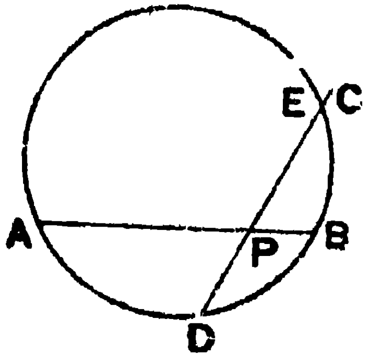


Fig. 1.

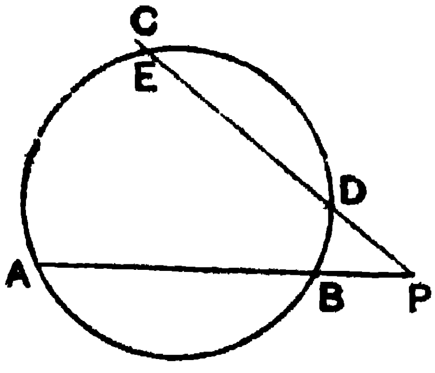


Fig. 2.

Let the straight lines AB, CD intersect at P, both internally in Fig. 1, or both externally in Fig. 2; and let the rect. AP,PB = the rect. CP,PD.

To prove that A, B, C, D *are concyclic.*

Construction. Let a circle be drawn through A, B and D.

Proof. If the circle does not pass through C, let it cut CD or CD produced at E.

Then because the chords AB, ED intersect at P;

∴ the rect. AP,PB = the rect. EP,PD.

But the rect. AP,PB = the rect. CP,PD, [*Hyp.*

∴ the rect. EP,PD = the rect. CP,PD;

∴ EP = CP;

∴ E coincides with C;

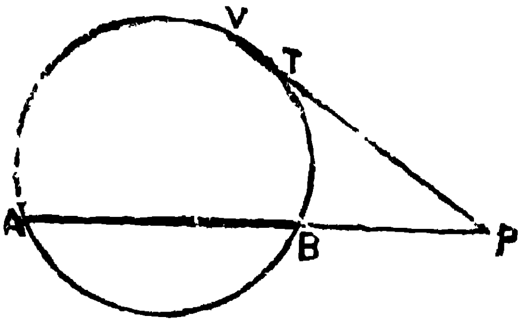
thus the circle through A, B and D also passes through C; that is, A, B, C, D are concyclic.

Q.E.D.

*THEOREM 13. [Euc. III. 37.]

[Converse of Theorem 11, Cor. 2.]

If a chord of a circle passes through a point outside the circle and from the point a straight line is drawn to meet the circle, and if the rectangle contained by the segments of the chord is equal to the square on the straight line, then this line touches the circle at the point where it meets the circle.



Let the chord AB pass through a pt. P outside the circle, and PT be drawn from P to meet the circle at T; and let $AP.PB = PT^2$.

To prove that PT *touches the circle at* **T**.

Proof. If PT does not touch the circle, then it will, if produced, meet the circle again at some pt. V.

Then because the chords AB, VT meet at P outside the circle;

$\therefore AP.PB = VP.PT.$ [IV.11.

But $AP.PB = PT^2$; [*Hyp.*

$\therefore VP.PT = PT^2$;

$\therefore VP = PT,$

that is, V coincides with T;

∴ PT when produced, does not meet the circle at any other point.

Hence PT touches the circle at T.

Q.E.D.

Theorem 13 may also be enunciated thus :

If from a point outside a circle, a secant is drawn and also a straight line to meet the circle, and if the rectangle, contained by the whole secant and the part of it outside the circle, is equal to the square on the line that meets the circle, then this line touches the circle.

EXERCISES LXVI.

[*Graphical and Theoretical.*]

1. Two chords **AOB**, **COD** interesect at **O** within a circle ; if **AO**$=4$ cm., **OB**$=3$ cm., and **OC**$=5$ cm., find **CD** and check the result by measurement.

2. From a point **A** outside a circle of radius 4", a straight line **ABC** is drawn through the centre cutting the circle at **B, C** ; if **AB** $=5$", find the length of the tangent from **A** to the circle. Verify by measurement.

5. **P** is a point outside a circle of radius 2·1" ; **PT** is a tangent from **P** whose length is 2·8" ; find the shortest distance of **P** from the circle.

4. **AB, AC** are tangents from **A** to a circle whose centre is **O** ; **OA** cuts **BC** at **P** ; if **OA**$=8$ cm., and **OP**$=3{\cdot}5$ cm., find the radius of the circle.

5. **PQ** is the chord of an arc of a circle whose diameter is 1·3" ; if **PQ**$=$ 1·2", find the height of the arc.

6. Two chords **AOB**, **COD**, 11 and 10 cm long respectively, intersect at **O** within a circle ; if **OC**$=4$ cm. find **OA**.

7. Prove the formula $t^2=(h+2r)$, where t denotes the length of the tangent to a circle from an external point, h the shortest distance of the point from the circle, and r the radius of the circle.

8. **Prove by Theorem 11 that the tangents to a circle from an external point are equal.**

9. **The common chord of two intersecting circles, if produced, bisects their common tangents.**

10. **If two circles intersect, the tangents drawn to the circles from any point in their common chord produced, are equal.**

11. **A, B, C, D** are four points in order in a straight line; find a point **O** in that line, such that

(i) $OA.OB = OC.OD$;

(ii) $OA.OC = OB.OD$;

and (iii) $OA.OD = OB.OC$.

12. If two circles intersect and two chords are drawn, one in each circle, to pass through any point in their common chord, or that chord produced, prove that the extremities of these chords are concyclic.

This Theorem is true when the the circles touch internally, or externally. Prove this case,

13. The altitudes **AM, BN, CR** of a triangle **ABC** intersect at **O**; prove that

(i) $AC.CN = BC.CM$;

(ii) $AN.AC = AO.AM = AR.AB$;

(iii) $AO.OM = BO.ON = CO.OR$.

14. **ABC** is a triangle right-angled at **A**; **AD** is perp. to **BC**; shew that $AB^2 = BD.BC$ and $AC^2 = CD.CB$.

15. From an external point **P**, tangents are drawn to a circle of centre **O** and **OP** cuts the chord of contact at **Q**; shew that

(i) $OP.PQ = (\text{tangent})^2$; (ii) $OP.OQ = (\text{radius})^2$.

16. **If three circles intersect one another, the three common chords are either concurrent or parallel.**

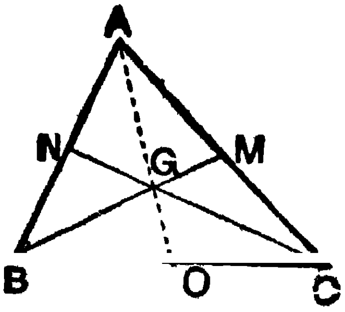
17. **P** is any point in a fixed straight line **AB**, and **AQ** is any chord of the circle described on **AP** as diameter. If **AQ** is produced to **R** so that $AQ.AR = AP.AB$, prove that **R** lies on a fixed straight line.

18. Circles are drawn through two fixed points; find the locus of a point from which the tangents to all these circles are equal.

19. Circles are drawn through two fixed points **A** and **B**, and from a fixed point in **AB** produced, tangents are drawn to them; prove that the locus of the points of contact of these tangents is a circle.

THEOREM 14.

The three medians of a triangle are concurrent.



Let ABC be a triangle; and let BM, CN be two of its medians intersecting at G.

Join AG, and produce it to meet BC in O.

To prove that AO *is the third median of the triangle.*

Proof. Because AM = MC,

∴ the △ ABM = the △ CBM,

also the △ AGM = the △ CGM [II. 3, *Cor.*

By subtraction, the △ AGB = the △ CGB,

Similarly, because AN = BN,

∴ the △ AGC = △ the BGC,

∴ the △ AGB = the △ AGC.

Now two equal △s AGB, AGC are on the same base AG and on opposite side of it.

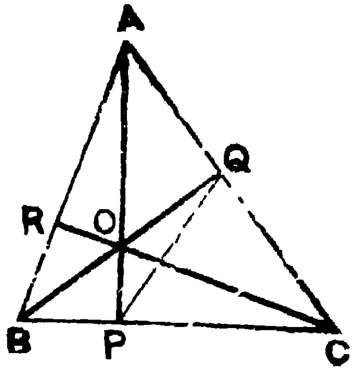
∴ AG bisects BC. [*App.* II. (5).

Q.E.D.

Note. We have already proved this Theorem [App. I, (8)].

THEOREM 15.

The three altitudes of a triangle are concurrent.



Let ABC be a triangle, and let AP, BQ be two of its altitudes intersecting at O.

Join CO, and produce it to meet AB in R.

To prove that CR *is the third altitude of the triangle* ABC.

Proof. Because the $\angle^s$ OPC, OQC are rt. $\angle^s$,
∴ OPCQ is a cyclic quadrilateral, [III. 15.
∴ the ∠OPQ = the ∠OCQ.
Again ∵ the $\angle^s$ APB, BQA are rt. $\angle^s$,
∴ ABPQ is a cyclic quadrilateral,
∴ the ext. ∠QPC = the int. opp. ∠BAQ.
[III. 14, *Cor.*

Now the ∠BAQ + the ∠OCQ = the ∠QPC + the ∠OPQ
= the ∠APC
= 1 rt. ∠.

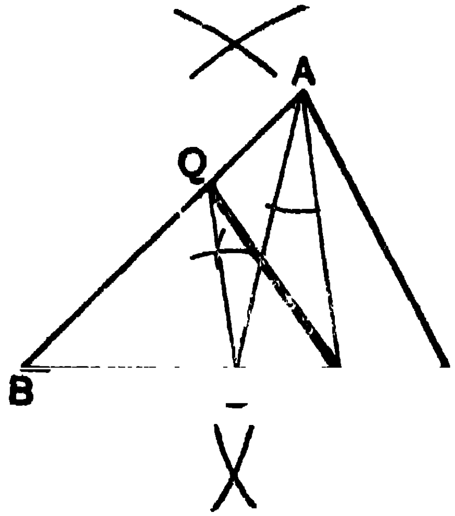
Then in the △ ARC, the $\angle^s$ RAC, RCA = 1 rt. ∠.
[*Proved.*
∴ the ∠ARC = 1 rt. ∠
that is, CR is the altitude of the △ABC.

Q.E.D.

Note. We have already proved this theorem. [App. I. (6) and App. III. (6)].

CONSTRUCTIONS

Problem 1. To bisect a triangle by a straight line drawn through a given point in one of its sides.



Let ABC be a given triangle and P a given point in one of the sides BC.

To bisect the triangle ABC *by a st. line drawn through* P.

Construction. Bisect BC at D.

Join AP and through D, draw DQ parl. to AP, meeting AB in Q. Join AD, PQ.

Then PQ bisects the triangle ABC.

Proof. Because BD = CD

∴ the △ABD = the △ACD in area.

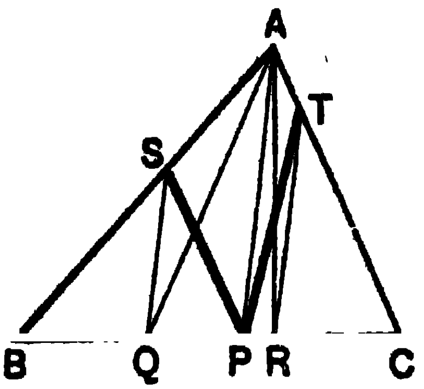
The △s DQP, DQA being on the same base DQ and of the same altitude are equal.

Add the △QDB;

then the △QPB = the △ABD

= ½ △ABC.

Problem 2. **To trisect a triangle by straight lines drawn through a given point in one of the sides.**



Let ABC be the given triangle and P a given point in one of the sides BC.

To trisect the triangle ABC *by st. lines drawn through* P.

Construction. Trisect BC at Q and R.

Join AP and through Q and R draw QS and RT parl. to AP.

Join PS, PT.

Then PS and PT trisect the triangle ABC.

Proof. Join AQ, AR.

Now the Δ^s PSQ, ASQ are on the same base SQ and between the same parallels SQ, AP ;

$\therefore$ the Δ PSQ = the Δ ASQ in area.

To each add the Δ SBQ,

$\therefore$ the Δ SBP = the Δ ABQ = $\frac{1}{3}$ the Δ ABC.

[$\because$ BQ = $\frac{1}{3}$ BC]

Similarly the Δ TCP = the Δ ARC = $\frac{1}{3}$ the Δ ABC ;

$\therefore$ the remaining fig. ASPT = $\frac{1}{3}$ the Δ ABC.

EXERCISES LXVII.

1. Through a given point in one of the sides of a triangle, draw a straight line cutting off a third, a fourth, a fifth, or any required, part.

2. Trisect a parallelogram by straight lines drawn through an angular point.

3. **Bisect a quadrilateral by a straight line drawn through a given angular point.**

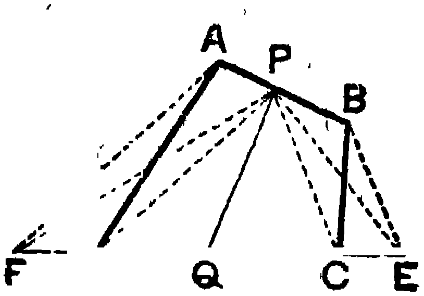
[Let **ABCD** be the given quadl and **A** the given angular pt. Join **AC**. Through **B**, draw **BM** ‖ to **AC**, meeting **DC** produced in **M**. Join **AM** and bisect **DM** in **X**. Then **AX** bisects the quadl. **ABCD**.]

4. Through an angular point of a quadrilateral, draw a straight line cutting off a third, a fourth, a fifth, or any required, part.

5. **Bisect a quadrilateral by a straight line drawn through a given point in one of the sides.**

Let **ABCD** be the given quadl. and **P** a given pt. in **AB**.

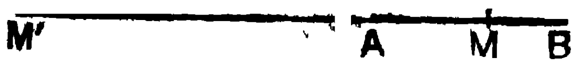
Const. Join **PC**, **PD**. Through **B**, draw **BE** ‖ to **PC**, and through **A**, draw **AF** ‖ to **PD**, meting **DC** produced both ways in **E**, **F**. Join **PE**, **PF** and bisect **EF** in **Q**. Join **PQ**.



Then **PQ** bisects the quadl. **ABCD**.

MEDIAL SECTION

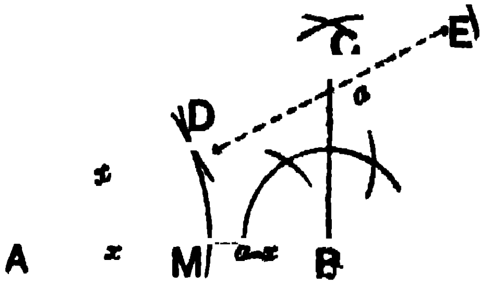
Definition. A straight line is said to be divided in **medial section** when the rectangle contained by the whole line and one segment is equal to the square on the other segment.



Thus (i) if $AB.BM = AM^2$, the line AB is divided *internally* in medial section at **M**.

(ii) if $AB.BM' = AM'^2$, the line AB is divided *externally* in medial section at **M'**.

***Problem 3. To divide a given straight line internally in medial section.**



Let AB be the given st. line.

To divide AB *internally at* M, *so that* AB.BM *may be equal to* AM^2.

Construction. Draw BC at rt. angles to AB, and make BC$=\frac{1}{2}$AB.

Join AC. With centre C and radius CB, draw a circle, cutting AC in D and AC produced in E. With centre A and radius AD, draw an arc, cutting AB in M.

Then AB is divided internally in medial section at M.

Proof. Let AB$=a$ and AM$=x$; then BM$=a-x$, AD$=x$, DE$=a$, AE$=x+a$.

From the right-angled △ ABC,

$$AB^2=AC^2-BC^2=(AC-BC)(AC+BC)$$

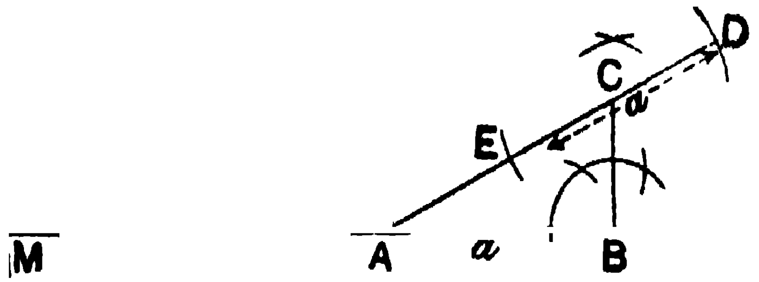
$$=(AC-CD)(AC+CE)=AD.AE\ ;$$

thus $$a^2=x(x+a)=x^2+ax,$$

$$\therefore\ a^2-ax=x^2,\text{ or } a(a-x)=x^2\ ;$$

that is, $$AB.BM=AM^2.$$

Problem 4. To divide a given straight line externally in medial section.



Let AB be the given straight line.

To divide AB *externally at* M, *so that* AB.BM *may be equal to* AM^2.

Construction. Draw BC at right angles to AB and make $BC = \frac{1}{2}AB$.

Join AC. With centre C and radius CB, draw a circle cutting AC in E and AC produced in D. With centre A and radius AD, draw an arc, cutting BA produced in M.

Then AB is divided externally in medial section at M.

Proof. Let $AB = a$ and $AM = x$; then $BM = x + a$, $AD = x$, $DE = a$, and $AE = x - a$.

From the right-angled △ ABC,

$$AB^2 = AC^2 - BC^2 = (AC + BC)(AC - BC)$$
$$= (AC + CD)(AC - CE) = AD.AE\ ;$$

thus $$a^2 = x(x - a) = x^2 - ax,$$
$$\therefore\ a^2 + ax = x^2,$$
$$\text{or } a(a + x) = x^2,$$
that is, $$AB.BM = AM^2.$$

EXERCISES LXVIII.

1. If a straight line is divided in medial section, the rectangle contained by the segments is equal to the rectangle contained by the sum and difference of the segments.

2. **AB** is divided internally in medial section at **C**; if from **AC**, the greater segment, a part **AD** is cut off equal to **BC**, prove that **AC** is divided internally in medial section at **D**.

3. If a straight line is divided internally in medial section, shew that the squares on the whole line and on the smaller part are equal to three times the square on the other part.

4. A straight line 5 cm. long is divided internally in medial section; find the lengths of the segments correct to two decimal places.

5. If the line in the last example is divided externally in medial section, find the lengths of the segments correct to two decimal places.

6. In a right-angled triangle, if the square on one side is equal to the rectangle contained by the hypotenuse and the other side, then the hypotenuse is cut in medial section by the perpendicular on it from the right angle.

7. On a given straight line as hypotenuse, construct a right-angled triangle such that the square on one side may be equal to the rectangle contained by the other side, and the hypotenuse.

8. Prove that **AB** is divided *internally* in medial section at **M**, from the following construction :—On **AB**, describe the square **ABCD**; bisect **AD** at **E**; join **EB**; with centre **E** and radius **EB**, draw a circle, cutting **EA** produced at **F**; with centre **A** and radius **AF** draw a circle, cutting **AB** at **M**; through **M**, draw **LMN** || to **FD** meeting **CD** at **N**, and through **F**, draw **FL** || to **AM** or **DN**. Then **M** is the pt. of division required.

9. Prove that **AB** is divided *externally* in medial section at **M**, from the following construction :—On **AB**, describe the square **ABCD**; bisect **AD** at **E**; join **EB**; with centre **E** and radius **EB**, draw a circle, cutting **ED** produced at **F**; with centre **A** and radius **AF**, draw a circle cutting **BA** produced at **M**; through **M**, draw **MNL** || to **AF**, meeting **CD** produced at **N**; through **F**, draw **FL** || to **AM** or **DN**. Then **M** is the required point of division.

Problem 5. To construct a square equal to a given rectangle.

First Method

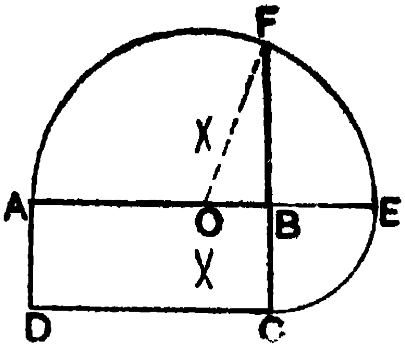


Fig. 1.

Let ABCD be the given rectangle.

To construct a square equal to ABCD.

Construction. Produce AB to E, making BE equal to BC.

Bisect AE at O; and with centre O and radius OA or OE, draw a semi-circle.

Produce CB to cut the semi-circle at F.

Then BF is a side of the required square.

Proof. Join OF.

$$\begin{aligned}\text{Now rect. } AC &= AB.BC = AB.BE\\ &= (AO+OB)(OE-OB)\\ &= (OE+OB)(OE-OB), \because AO = OE,\\ &= OE^2 - OB^2 = OF^2 - OB^2\\ &= BF^2, \because \text{ the } \angle OBF \text{ is a rt. angle.}\end{aligned}$$

$\therefore$ the rect. ABCD = the sq. on BF.

Second Method

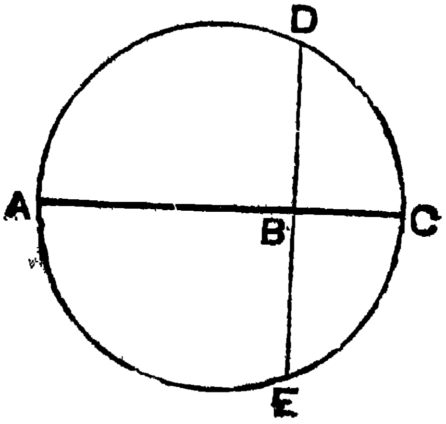


Fig. 2.

Construction. Take a st. line AC and let AB, BC be equal to the adjacent sides of the given rectangle.

On AC as diameter, draw a circle and let DBE be a chord perp. to AC at B.

Then BD is a side of the required square.

Proof. Because the diameter AC is perp. to the chord DE ;

∴ BD = BE. [III. 1.

And the rect. AB,BC = the rect. DB,BE

= the sq. on BD. [IV. 11.

Note. From the above construction, it is easy to see that

To construct a square equal to a given rectilineal figure

[Euc. II. 14.]

we have

(i) to construct a triangle equal to the given figure [II. Prob. 2]

(ii) to construct a rectangle equal to the triangle [II. Prob. 1, *Note.*] and

(iii) to proceed as in the above construction.

Cor. *If from any point on a semi-circle a perpendicular be drawn to the diameter, the square on this perpendicular = the rectangle contained by the segments of the diameter.*

Since the angle formed by two straight lines joining any point on a semi-circle to the extremities of the diameter is a right angle [(VIII), 10], the angle AFE (Fig. 1) is a right angle. Hence the theorem :—

In a right angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the square on this perpendicular = the rectangle contained by the segments of the hypotenuse.

EXERCISES LXIX.

[*Numerical and Graphical.*]

1 The adjacent sides of a rectangle are 8 cm. and 4·5 cm. ; construct a square of equal area and find the side by measurement. Verify by calculation.

2. A square is described equal in area to a rectangle whose sides are 4″ and 2·6″. Measure its side and verify by calculation.

3. On a line 3·2″ in length, construct a rectangle whose area is equal to that of a square which has a side of 1·2″. Find the other side of the rectangle.

4. Find graphically the side of a square equal to a rectangle whose area is 12·5 sq. cm. and one of whose sides is 2·5 cm. Verify by calculation.

[*Theoretical.*]

5. **On a given straight line, construct a rectangle equal to a given square.**

6. Having given the perimeter, construct a rectangle equal in area to a given square.

7. Having given the perimeter, construct a rectangle equal in area to a given rectangle.

8. Having given the difference of two adjacent sides, construct a rectangle equal in area to a given square.

9. Of all rectangles of given area, the square has the least perimeter.

10. Of all rectangles of given perimeter, the square has the greatest area.

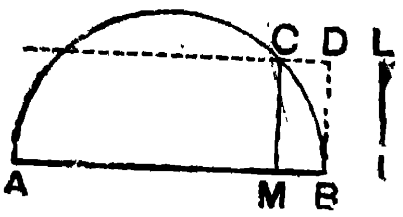
11. Divide a given straight line into two parts, so that the rectangle contained by the parts may be equal to a given square. When is this problem impossible ?

12. Produce a given straight line so that the rectangle contained by the whole line thus produced and the part produced may be equal to a given square.

The following two Problems which depend on Problem 5, enable us to *solve quadratic equations graphically.*

PROBLEM 5(*a*). **To divide a given straight line *internally* so that the rectangle contained by the segments may be equal to a given square.**

Let AB be the given line and L a side of the given square.



Construction. On AB, draw a semi-circle ACB. Draw BD ⊥ to AB and make BD = L. Draw DC ∥ to AB, cutting the semi-circle in C. Draw CM ⊥ to AB.

Then M is the required point of division.

Proof. BDCM is a rectangle ; ∴ CM = BD = L.

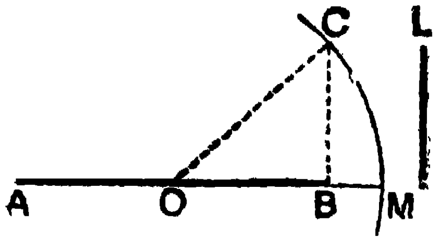
And C is a pt. on the semi-circle ACB ;

$\therefore CM^2 = AM.MB$ [IV. Prob. 5, Cor.

Hence $L^2 = AM.\ MB$.

PROBLEM 5(*b*). **To divide a given straight line *externally* so that the rectangle contained by the segments may be equal to a given square.**

Let AB be the given line and L a side of the given square.



Construction. Draw BC ⊥ to AB and make BC equal to L. Bisect AB at O and with centre O and radius OC, draw a circle, cutting AB produced at M.

Then M is the required point of division.

Proof. $CB^2 = OC^2 - OB^2$

$= (OC + OB)(OC - OB)$

$= (OM + OA)(OM - OB)$

$= AM.MB.$

Hence $L^2 = AM.MB$, ∵ BC = L (const).

Question. In Probs. 5(*a*) and 5(*b*), is there any limit to the magnitude of L ?

GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS

Ex. 1. To solve the equations $x+y=7$ and $xy=9$

Here we have to find two numbers x and y whose sum is 7 (say 7 cm.) and product 9, *i.e.*, 3^2 cm.

Take a line **AB** 7 cm. and **L** 3 cm. in length. Divide **AB** internally as in Prob. 5 (*a*) at **M**, so that $\mathbf{AM.MB=L^2}$.

Thus $\mathbf{AM}=x$, $\mathbf{MB}=y$. By measurement, $\mathbf{AM}=5{\cdot}3$ cm. and $\mathbf{MB}=1{\cdot}7$ cm.

Ex. 2. To solve the equation $x^2-10x+16=0$.

From the Theory of Quadratic Equations, we know that the sum of the roots $=10$ and their product $=16$ or 4^2. [See *Matric. Algebra*, Art. 307]

Thus we have to find two numbers whose sum is 10 and product 4^2 units. Hence as in Ex. 1, we divide **AB** at **M**, so that $\mathbf{AM.MB=L^2}$.

Thus **AM, MB** are the roots required. Measure **AM, MB** ; thus $\mathbf{AM}=8$ units and $\mathbf{MB}=2$ units.

Ex. 3. To solve the equation $x^2-5x-36=0$

Since the product of the roots is -36, the roots are of opposite signs. Also their algebraic sum $=5$.

Hence take $\mathbf{AB}=5$ units and $\mathbf{L}=6$ units (*i.e.*, $\sqrt{36}$ units). Divide **AB** externally as in Prob 5(*b*) at **M**, so that $\mathbf{AM.MB=L^2}$. Thus the roots are $\mathbf{AM}=9$ units and $\mathbf{MB}=-4$ units.

Note. In an equation $x^2-x+14=0$, where the last term 14 is not a perfect square, find $\sqrt{14}$ as in arithmetic and then proceed as before.

If in an equation x^2 has a coefficient as in $2x^2-11x-12=0$, reduce it to the form $x^2-\frac{11}{2}x-6=0$, by dividing by the coefficient of x^2.

EXERCISES LXIX(*a*).

1. Find by a geometrical construction two numbers whose sum $=8$ and product $=9$.

2. Find by a geometrical construction two number whose difference $=3$ and product $=9$

Solve graphically (correct to one decimal place) the equations

3. $x+y=15$, $xy=36$.
4. $x-y=6$, $xy=16$.
5. $x+y=9$, $xy=16$.
6. $x-y=3$, $xy=9$.
7. $x^2-13x+36=0$.
8. $x^2-6x+4=0$.
9. $x^2-12x+36=0$.
10. $x^2-16x+25=0$.
11. $x^2-6x-16=0$.
12. $x^2-5x-9=0$.
13. $x^2-7x+10=0$.
14. $4x^2-36x+9=0$.

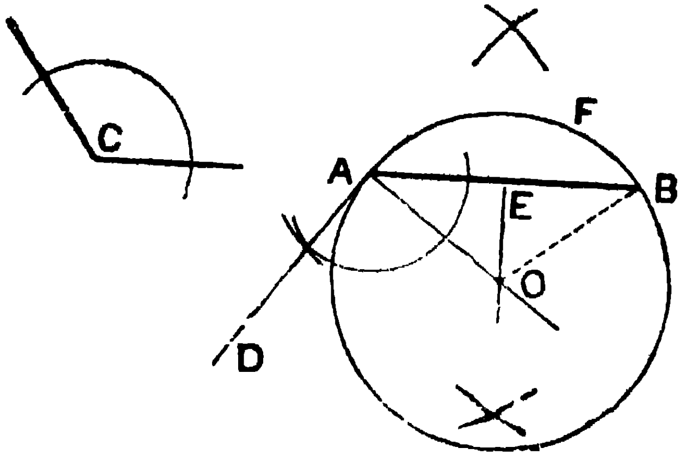
Problem 6. On a given straight line to construct a segment of a circle containing an angle equal to a given angle. [Euc. III. 33.]

Let AB be the given st. line and C the given angle.

On AB *to construct a segment of a circle containing an angle equal to the* ∠C.

Construction. At A in AB, make the ∠BAD = the ∠C.

From A, draw AO perpendicular to AD.



Bisect AB at rt. angles by EO, cutting AO at O.

Join OB.

Then every pt. in EO is equidistant from A and B ;

∴ OA = OB.

With centre O and radius OA or OB, draw a circle.

Then the segment AFB contains an angle = the ∠C.

Proof. Because AD is perp. to the radius AO ;

∴ AD is the tangent at A ; [III. 13, *Cor.* 1.

∴ the ∠DAB = the angle in the alt. segment AFB.

But the ∠DAB = the ∠C. [*Const.*

∴ the segment AFB contains an angle = the ∠C.

EXERCISES LXX.

[*Practical and Theoretical.*]

1. **Prove the following alternative construction of Problem 6**:—Let **AB** be the given st. line and **C** the given angle. Divide the supplement of the ∠**C** into any two parts ; at **A** make the ∠**BAP**=one part and at **B** make the ∠**ABP**=the other part ; circumscribe a ⊙ about the △**ABP**. Then **APB** is the segment required.

2. On a base of 1·5″, draw five △s on each side of the base, each having a vertical angle of 90°. What is the locus of their vertices ?

3. On a base of 4 cm., draw five △s on one side of the base, each having a vertical angle of 60°. What is the locus of their vertices ?

4. **The locus of a point at which a given straight line subtends a constant angle is an arc of a circle whose chord is the given line.**

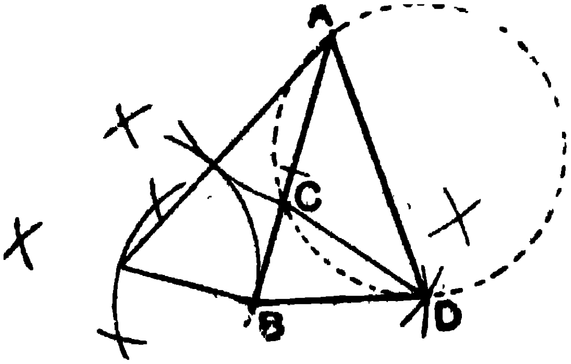
5. Construct an isosceles triangle on a given base and having a given vertical angle.

6. On a base of 2″, construct a triangle with a vertical angle of 53°, having (i) a side=2·4″, (ii) an altitude=1·9″ and (iii) a median bisecting the base=2.″

7. **Construct a triangle with a given base, a given vertical angle and (i) a given altitude, (ii) a given side, (iii) a given area, (iv) a given median bisecting the base, and (v) the point in the base where the perpendicular from the vertex falls.**

8. Construct a triangle having given the base, the vertical angle, and the point where the internal bisector of the vertical angle cuts the base.

Problem 7. To construct an isosceles triangle having each of the angles at the base double of the vertical angle.



Construction. Take any st. line AB.

Divide AB at C, so that $AB.BC = AC^2$. [IV. *Prob.* 3.

With centres B and C, and radius equal to AC, draw two arcs intersecting at D.

Join AD.

Then ABD is the required triangle.

Proof. Circumscribe a circle about the △ ACD.

Now $AB.BC = AC^2 = BD^2$; [*Const.*

∴ BD touches the ⊙ ACD at D. [IV. 13.

∴ the ∠BDC = the ∠CAD in the alt. segment. [III. 17.

And because CA = CD;

∴ the ∠CDA = the ∠CAD. [*Const.*

∴ the ∠BDA = the ∠BDC + the ∠CDA = 2∠CAD.

Again because DB = DC; [*Const.*

∴ the ∠DBC = the ∠DCB = the ∠CAD + the ∠CDA
= 2∠CAD.

∴ the ∠DBC = the ∠BDA;

that is, the ∠ABD = the ∠ADB;

∴ AB = AD.

Hence ABD is an isosceles triangle, having each of the ∠s ABD, ADB double of the ∠A.

EXERCISES LXXI.

[*Theoretical.*]

N.B. The following Exercises refer to the Figure of Problem 7.

1. Find in degrees the magnitudes of the angles **BAD, ABD, ADB ACD.**

2. Shew that there is another triangle which has the same property as the △**ABD.**

3. If the circle with centre **A** and radius **AB** or **AD** cuts the circle **ACD** again at **E,** shew that **DE=AC.**

4. Shew that the circle **ACD** is equal to the circumcircle of the △ **ABD.**

5. In the fig. of Ex. 3, prove that **CE** is parallel to **BD,** and **AE** to **CD.**

6. If the tangent at **A** to the circle **ACD** cuts **DB** produced at **F,** shew that the △**FAD** possesses the same property as the △**ABD,** and the △**ABF** the same property as the △**ACD.**

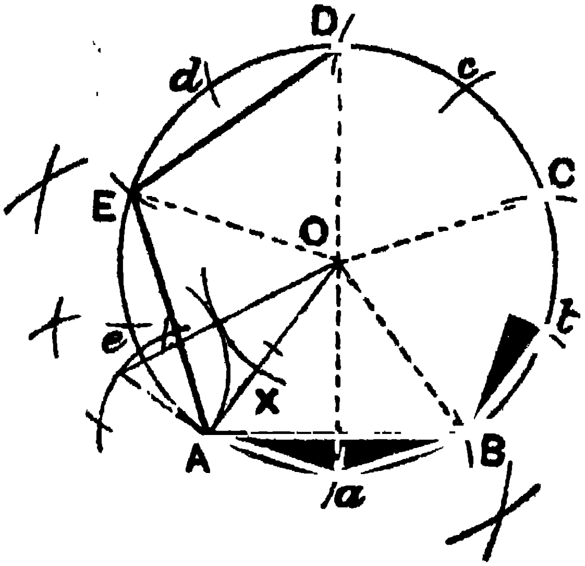
7. In the fig. of Ex. 3, if **AE** and **BD** be produced to meet at **G,** prove that the △**GAB** is of the same kind as the △**ABD.**

8. Prove that the circumcentre of the △**BCD** is the middle point of the arc **CD.**

9. Prove that the circumcentre of the △**BCD** is equidistant from the incentres of the △s **ABD, BCD.**

10. If **AK** be a diameter of the circle **ACD** cutting it at **K,** shew that **DK** is a circum-radius of the △**BCD.**

Problem 8. **To inscribe a regular pentagon in a given circle.** [Euc. IV. 11.]



Let O be the centre of the given circle, in which a regular pentagon is to be inscribed.

Construction. Take any radius OA; and divide it at X, so that $OA.AX = OX^2$.

Place chords A*a*, *a*B in the circle; each equal to OX.

Join AB; and place chords BC, CD DE, each equal to AB, round the circle. Join EA.

Then ABCDE is the pentagon required.

Proof. Join O*a*, OB, OC, OD, OE.

Because the arc A*a* = the arc *a*B; ∴ the ∠AOB = 2∠AO*a*

and by Prob. 7, the ∠AO*a* = $\frac{1}{5}$ of 2 rt. angles.

Hence the ∠AOB = $\frac{1}{5}$ of 4 rt. angles;

∴ AB is the side of a regular pentagon inscribed in the circle.

Now because chords AB, BC, CD, DE are equal;

∴ the arcs AB, BC, CD, DE are equal;

∴ the ∠AOB = the ∠BOC = the ∠COD = the ∠DOE.

And because the sum of these angles $=\frac{4}{5}$ of 4 rt. angles ;

$\therefore$ the $\angle$ EOA $=\frac{1}{5}$ of 4 rt. angles $=$ the $\angle$ AOB.

Thus the arc AE $=$ the arc AB ;

and $\therefore$ the chord AE $=$ the chord AB.

$\therefore$ the fig. ABCDE is equilateral.

Again because the arc BCDE $=$ the arc AEDC ;

$\therefore$ the $\angle$ A $=$ the $\angle$ B.

Similarly the $\angle^{s}$ C, D and E are each equal to the $\angle$ A.

$\therefore$ the fig. ABCDE is equiangular.

Hence the pentagon ABCDE is regular ; and it has been inscribed in the given circle.

Problem 9. To inscribe a regular decagon in a given circle.

Let O be the centre of the given circle [see Fig. of Problem 8], in which a regular decagon is to be inscribed.

Construction. Draw any radius OA ; and divide it at X, so that $OA.AX=OX^2$.

In the circle, place nine chords A*a*, *a*B, B*b*, *b*C,……each equal to OX. Join *e*A.

Then A, *a*, B..., *e*, are the vertices of a regular decagon inscribed in the circle.

[Supply Proof.]

Note. A *polygon of* n *sides* is sometimes written as *n*-gon ; so a pentagon is written as 5-gon, a hexagon as 6-gon, an octagon as 8-gon ; and so on.

CONSTRUCTION OF CIRCLES SATISFYING GIVEN CONDITIONS

The following Theorems (in thick type) regarding Loci most of which have already been given, are important in connection with the Construction of Circles.

Ex. 1. Take a pt. **A** and draw five circles each having a radius of 1·5 cm., to pass through **A**. Draw the locus of the centres of these circles, and see if it is a ⊙ whose centre is **A** and radius 1·5 cm. Thus

(1) The locus of the centre of a circle of given radius which passes through a given point is a circle whose centre is the given point and radius = given radius.

Ex. 2. Draw a circle of radius 2 cm.: draw five circles each having a radius of 1·5 cm. to touch it externally. Draw the locus of the centres of the circles and see if its radius = 3·5 cm.

Again, draw a circle of radius 3 cm.; draw five circles each having a radius of 1 cm. to touch it internally. Draw the locus of their centres and see if its radius = 2 cm. Thus

(2) The locus of the centre of a circle of given radius which touches a given circle is a concentric circle whose radius = sum or difference of the radii, according as the contact is external or internal.

Ex. 3. Take a straight line **AB**; draw four circles each of radius 1 cm. to touch it on one side. Draw the locus of their centres. Draw four circles of the same radius on the other side of **AB** to touch it; and draw the locus of the centres of these circles. See that the locus consists of a pair of st. lines parallel to **AB** and at a distance of 1 cm. on either side of **AB**. Thus

(3) The locus of the centre of a circle of given radius which touches a given straight line is a pair of lines parallel to the given line on either side of it and at a distance = given radius.

Ex. 4. Let **A** be a given point on the circumference of given circle (**C**). Draw five circles of radii, 1, 2, 3, 4 and 5 cm. to touch (**C**) at **A**. Shew that **CA** is the locus of their centres. Hence

(4) The locus of the centre of a circle which touches a given circle at a given point is the join of the given point to the centre of the given circle.

Ex. 5. Take a point **P** a line **AB**. Draw three circles of radii 1, 2 and 3 cm. to touch **AB** at **P** on one side, and five cirles of radii 4, 4·5, 2·5, 3 and 1·5 cm. to touch **AB** at **P** on the other side. Draw the locus of the centres of these circle. Thus

(5) The locus of the centre of a circle which touches a given line at a given point is the perpendicular to the line at the given point.

Ex. 6. Take two points **A** and **B**, 4 cm. apart. Through **A** and **B**, draw seven circles (some on one side of **AB** and some on the other) of radii 2, 3, 3·5, 3·8, 4, 4·5 and 5 cm. Draw the locus of their centres. Hence

(6) The locus of the centre of a circle which passes through two given points is the perpendicular bisector of the join of the given points.

Ex. 7. Take two st. lines **AOB**, **COD** intersecting at **O**. Draw a number of circle of different radii to touch them in each of the four compartments into which they divide the plane of the paper. Draw the locus of their centres. Thus

(7) The locus of the centre of a circle which touches two intersecting straight lines is the pair of bisectors of the angles between them.

Ex. 8. Draw two parallel lines **AB**, **CD**, 7 cm. apart. Draw a number of circles to touch them ; and then draw the locus of their centres. What is the magnitude of the radius of each circle ? Hence

(8) The locus of the centre of a circle which touches two parallel straight lines is a line parallel to the given lines and midway between them.

Ex. 9 Let **AB** be a given st. line and **P** a fixed point outside it. Draw **PM** perpendicular to **AB** and produce it to Q making **MQ=PM**. Then Q is a fixed point. Take any point **O** in **AB**, and draw a circle with centre O and radius **OP**. This circle shall pass through **Q**. It with other points in **AB** as centres, circles are similarly described, they shall all pass through **Q**. Hence

(9) All circles that pass through a fixed point and have their centres in a fixed straight line, pass also through a second fixed point, *viz.*, its image.

A circle can be drawn if the position of centre and the magnitude of its radius are known. The centre can often be found by the method of the Intersection of Loci. Thus *two* conditions, each giving a locus, are required to find the centre. To find the radius we want a *third* condition. Hence *three* conditions are necessary to draw a circle.

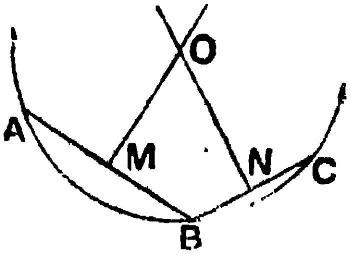
The student who has made himself familiar with the above Loci (in bold type), will find little difficulty in constructing circles under given conditions.

Ex. 1. Draw a circle to pass through three given points.

Let **A**, **B**, **C** be three given points.

To draw a circle to pass through **A**, **B** and **C**.

Construction. Since the required circle is to pass through **A** and **B**, its centre is equidistant from **A** and **B.** Thus the locus of the centre is **MO**, the perp. bisector of **AB**.



Similarly, the locus of centre of the required circle is **NO**, the perp. bisector of **BC**.

Hence **O**, the pt. of intersection of **MO**, **NO**, is the centre of the required circle.

The radius is clearly **OA**, **OB**, or **OC**, for **OA**=**OB**=**OC**.

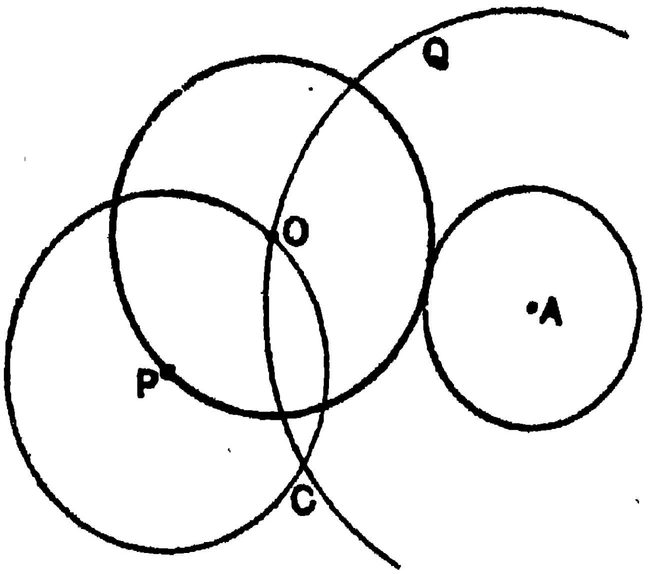
∴ the circle described with centre **O** and radius equal to **OA**, **OB** or **OC** is the required circle.

The construction fails when **MO** and **NO** do not intersect ; that is, when **A**, **B** and **C** are in the same straight line.

Ex. 2. To draw a circle of given radius (say 1·5 cm.) to pass through a given point and touch a given circle (of radius 1 cm. say).

Let **P** be the given pt. and (**A**) the given circle.

To draw a circle of radius 1·5 cm. which shall pass through **P** *and touch* (**A**).



By (1), the locus of centre of required ⊙ is a ⊙ whose centre is **P** and radius 1·5 cm. Let (**P**) be that ⊙.

By (2), the locus of centre of required ⊙ is a ⊙ whose centre is **A** and radius (1 + 1·5) cm.

Draw this ⊙ and let it be **COQ**, intersecting the ⊙ (**P**) at **O**.

Then **O** is the centre of required ⊙.

And its radius is given *viz.*, 1·5 cm.

With centre **O** and radius 1·5 cm., draw a ⊙ which is the required ⊙.

Since **COQ** intersects (**P**) at another pt. **C**, **C** is the centre of another circle which shall pass through **P** and touch (**A**).

Thus there are *two* solutions of the Problem.

Question. What are the three conditions for drawing the circle (**O**) ?

EXERCISES LXXII.

[*Theoretical*]

1. Draw a circle to pass through two given points and have its centre in a given straight line. When is the problem impossible ?

[One locus is given ; see (6).]

2. Draw a circle of given radius to pass through a given point and have its centre in a given straight line.
[One locus is given ; see (1).]

3. Draw a circle with a given centre to touch a given circle. How many solutions are there ? [See III. 16A].

4. Draw a circle to touch a given circle, at a given point and have its centre in a given straight line. [See (4).]

5. Draw a circle of given radius (1·5 cm.) to pass through a given point (distant 2·6 cm. from given line) and touch a given straight line. Shew that there are in general two solutions. When is there one solution ? When is the solution of the general problem impossible ? [See (1) and (3).]

6. Draw a circle of given radius to pass through a given point and touch a given circle. How many solutions ? [See (1) and (2).]

7. Draw a circle to pass through a given point and touch a given straight line at a given point. When does the construction fail ? [See (5) and (6).]

8. Draw a circle to pass through a given point and touch a given circle at a given point. [See (4) and (6).]

9. Draw a circle of given radius to pass through two given points. When does the construction fail ? [See (1) and (6).]

10. Draw a circle of given radius to touch a given circle and a given straight line. [See (2) and (3).]

11. Draw a circle of given radius to touch a given circle at a given point. [See (1) and (4), or (2) and (4).]

12 Draw a circle to touch two given straight lines and have its centre in a third given straight line. When is the solution impossible ? [See (7)]

13. Draw a circle of given radius to touch two given intersecting straight lines. [See (3) and (7).]

14. Draw a circle of given radius to touch two given circles. The given circles may be external to each other or one inside the other. Give the construction for both solutions. [See (2).]

15. Draw a circle of given radius to touch a given straight line and have its centre in another given straight line.
[One of the loci is given ; see (3).]

16. Draw a circle of given radius to touch a given circle and have its centre in a given straight line In general, two solutions. When only one ? [See (2).]

17. Draw a circle to touch each of three given straight lines (i) when no two of which are parallel, and (ii) when not more than two of which are parallel.

18. On the same side of a given straight line, draw two circles of given radii to touch one another and the given line. [See (3) and III. 16A.)

Miscellaneous Examples.

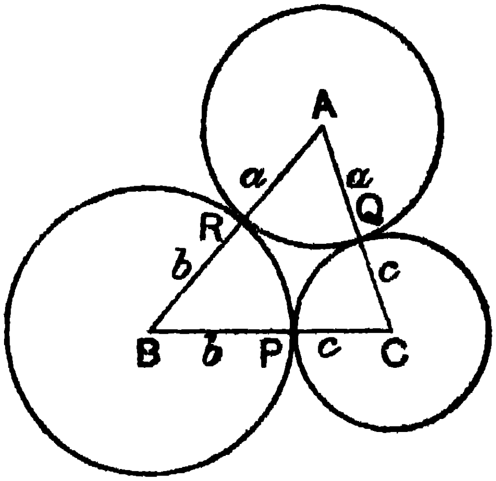
EXERCISES LXXIII.

1. **Draw three circles of given radii to touch one another externally.**

Let a, b and c be the given radii.

[**Analysis.** Suppose **(A)**, **(B)** and **(C)** be the required circles, their centres being **A**, **B** and **C** respectively. Let the circles touch at **P**, **Q** and **R**. Join **AB**, **BC** and **CA**. Then **BC** passes through **P**, **CA** through **Q** and **AB** through **R**. Thus **ABC** is a △, and $BP=BR=b$, $CP=CQ=c$ and $AQ=AR=a$.

Hence we have the following]



Construction. Construct a △**ABC**, whose side $AB=a+b$, $BC=b+c$ and $CA=c+a$.

With centres **A**, **B** and **C**, and radii a, b and c respectively, draw circles **(A)** **(B)** and **(C)**, meeting at **P**, **Q** and **R**.

Then these are the required circles.

Proof. Because $AB=a+b$, and $AR=a$, $BR=b$;

∴ circles **(A)** and **(B)** meet at one pt. **R** on the line of centres ;

∴ they touch one another at **R**. [III. 16 A.]

Similarly **(B)** and **(C)** touch at **P**, and **(C)** and **(A)** at **Q**.

2. Draw three circles of given radii—two of which shall touch each other externally and the third circle internally.

[*Hint.* If a, b, c are radii of required ⊙s, construct a △ whose sides are $AB=a+b$, $BC=c-b$ and $CA=c-a$.]

3. With three given points as centres, draw three circles to touch one another externally.

[*Hint.* See III. Prob. 5, and Ex. 1.]

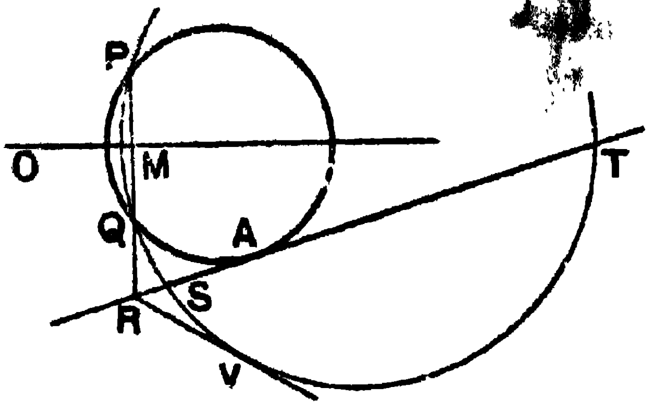
4. With three given points as centres, draw three circles—two of which shall touch each other externally and the third circle internally.

[*Hint.* See III. Prob. 6, and Ex. 2.]

5. **Draw a circle to touch each of three given equal circles. How many solutions ?**

6. Draw a circle to pass through a given point, to touch a given straight line and to have its centre on another given straight line.

Let **P** be the given point, **RT** the given st. line which the circle is to touch and **OM** the st. line which contains the centre of reqd. ⊙.



[**Analysis.** Suppose **APQ** to be the required circle. Because the circle passes through **P** and has its centre in **OM** ;

∴ it also passes through **Q**, the image of **P** in **OM**. [See (9)].

Produce **PQ** to cut **RT** in **R**.

Then since **RA** is a tangent ; ∴ $RP.RQ = RA^2$.

Draw a circle through **P** and **Q** to cut **RT** in **S** and **T**.

Then if **RV** be a tangent to this circle, $RP.RQ = RV^2$.

Thus $RA = RV$.

Hence the]

Construction. Draw **PM** perp. to **OM** ; produce **PM** to Q, and make $MQ = PM$.

Produce **PQ** to meet **RT** in **R**.

Draw a ⊙ through **P** and **Q** to cut **RT** in **S** and **T**.

Let **RV** be a tangent to this circle. From **RT**, cut off $RA = RV$.

Draw a circle through **P**, **Q** and **A**.

Then **APQ** is the required circle.

Proof. Because $RP.RQ = RV^2 = RA^2$;

∴ **RT** is a tangent to the ⊙ **APQ** at **A**.

Also **OM** is the perp. bisector of the chord **PQ** ;

∴ **OM** passes through the centre.

Note. Since **A** may be taken on either side of **R**, there is another solution.

7. **Draw a circle to pass through a given point, to touch a given circle and to have its centre on a given straight line.**

Let **P** be the given pt., **(B)** the given ⊙ and **C** the st. line containing the centre of the reqd. ⊙.

Construction. Draw **PM** perp. to **C**; produce **PM** to **Q** and make **MQ=PM.** Draw a ⊙ to pass through **P** and its image **Q**; and to cut the circle **(B)** in **S** and **T.**

Let **PQ** and **TS** meet at **R.** From **R**, draw **RA** to touch **(B)** at **A.** Describe a ⊙ through **P, Q** and **A.**

Then **APQ** is the required circle.

Note. The Problem has *two* solutions. Why?

8. Draw a circle to touch a given straight line, to have its centre in another given straight line, and to pass through a given point in this line.

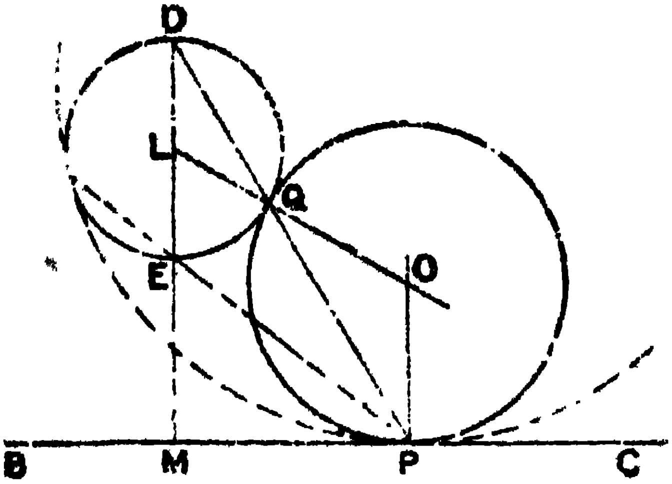
9. Draw a circle to touch a given circle, to have its centre in a given straight line and to pass through a given point in this line.

10. Draw a circle to touch a given straight line and a given circle and to have its centre on another given straight line. [See Ex. 6.]

11. Draw a circle to touch two given circles and to have its centre on a given straight line. [See Ex. 7.]

12. **Draw a circle to touch a given circle and a given straight line at a given point.**

Let **(L)** be the given circle, **BC** the given st. line and **P** the given pt. in it.



[**Analysis.** Let **(O)** be the circle which touches **(L)** at **Q** and **BC** at **P.**

Join **OP**; then **OP** is perp. to **BC**. Join **PQ** and produce it to cut (**L**) at **D**. Draw the diameter **DE**, and produce it to meet **BC** at **M**. Then **DM** is parl. to **OP** [(LVI), 10]; thus **DM** is perp. to **BC**.

Hence the following]

Construction. Draw the diameter **DE** perp. to **BC** meeting **BC** in **M**. Draw **PO** perp. to **BC**. Join **PD** cutting (**L**) in **Q**. Join **LQ** and produce it to cut **PO** in **O**. With centre **O** and radius **OP**, draw a circle.

Then this is the required circle.

Proof. Since **OP** and **DM** are perp. to **BC**; ∴ they are parallel. Hence the $\angle$**OPQ** = the alt. $\angle$**QDL** = the $\angle$**LQD** = the $\angle$**OQP**;

∴ **OP**=**OQ**;

∴ the circle (**O**) passes through **Q**.

Thus the circles (**L**) and (**O**) meet at a pt. **Q** on the line of centres; hence they touch one another at **Q**. [III. 16A.]

Note. By joining **PE** and producing it to cut (**L**), another solution is obtained. Thus the Problem has *two* solutions.

Question. Give a construction for the other solution.

13. **Draw a circle to touch a given straight line and a given circle at a given point.**

Let **BC** [see Fig., of Ex. 12] be the given st. line, (**L**) the given ⊙ and **Q** the given pt. on it.

Construction. Draw the diameter **DE** perp. to **BC**, meeting **BC** in **M**. Join **DQ** and produce it to cut **BC** in **P**. Draw **PO** perp. to **BC**. Join **LQ** and produce it to cut **PO** in **O**. With centre **O** and radius **OP**, draw a circle.

Then this is the required circle.

[Supply Proof.]

Note. By joining **QE** and producing it to cut **BC**, another solution is obtained. Thus the Problem has *two* solutions.

Question. Give a construction for the other solution.

For other problems on the Construction of Circles, see *App.* V. Section VII, "The Tangencies."

APPENDIX IV

The following theorems are very useful and should be carefully studied.

1. THEOREM. *The distance of the middle point of a finite straight line*

(i) *from a point of* **internal division** *is half the* **difference** *of the segments*; and

(ii) *from a point of* **external division** *is half the* **sum** *of the segments.*

Let AB be the given line, O its middle point, and M and P points of internal and external division, so that the respective segments are AM, MB and AP, PB.

(i) $AM - MB = AO + OM - MB$

$= OB + OM - MB$, if M is between O and B as in the fig.

$= OM + MB + OM - MB = 2OM$.

Similarly, if M is between A and O, it may be shewn that $MB - AM = 2OM$.

Hence always $2OM = AM \sim MB$.

(ii) $AP + PB = AB + BP + PB$

$= 2OB + 2PB = 2OP$, $\because OB + BP = OP$.

Thus $2OP = AP + PB$.

2. THEOREM. *If four points* A, B, C, D *are taken in order on a straight line, then*

$AC.BD = AB.CD + BC.AD$. [EULER'S THEOREM.]

By Theorem 1,

$AC.BD = AB.BD + BC.BD$ [AC is divided at B]

$= AB.CD + AB.BC + BD.BC$ [BD is divided at C]

$= AB.CD + BC.AD$.

Otherwise: Prove algebraically by taking

$AB = a$, $BC = b$ and $CD = c$.

3. **Theorem.** *If from the vertex of an isosceles triangle a straight line is drawn to the base, or the base produced, the rectangle contained by the segments of the base is equal to the difference of the squares on a side and on the line drawn from the vertex.*

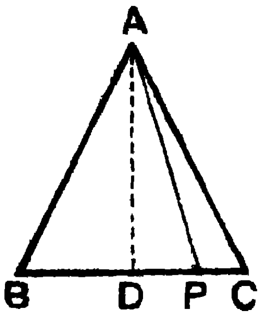


Fig 1.

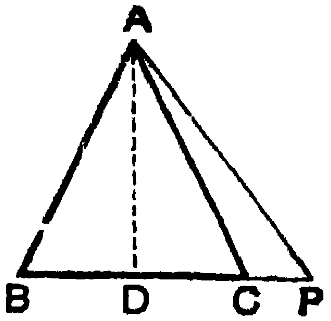


Fig 2.

Let ABC be an isosceles △, having AB = AC. Let a st. line AP be drawn from A to BC or BC produced.

To prove that BP.PC = the difference of the squares on AC and AP.

Construction. Draw AD perp. to BC.

Proof. Because AD is perp. to the base BC of an isosceles triangle; ∴ BD = DC.

$$\text{Now } BP.PC = CD^2 - DP^2 \text{ [Fig. 1] or } DP^2 - CD^2 \text{ [Fig. 2]}$$

$$= (AC^2 - AD^2) - (AP^2 - AD^2) \text{ [Fig. 1]}$$

$$\text{or } (AP^2 - AD^2) - (AC^2 - AD^2) \text{ [Fig. 2]}$$

$$= AC^2 - AP^2 \text{ or } AP^2 - AC^2$$

$$= AC^2 \sim AP^2.$$

Ex. 1. The base BC of an equilateral triangle ABC is produced to D, prove that $AD^2 = CD^2 + BD.BC$.

Ex 2. ABC is an equilateral triangle and D is any point in BC, prove that $AD^2 = BD^2 + DC^2 + BD.DC$.

Ex. 3. ABC is an equilateral triangle, BC is produced to D, so that $BD.DC = BC^2$; prove that $AD^2 = 2AB^2$.

4. **THEOREM.** *In any triangle, the difference of the squares on the two sides is equal to twice the rectangle contained by the base and the projection on the base of the median that bisects the base.*

Let ABC be a triangle, of which BC is the base. [See Figs. of IV. 10.]

Let AD be the median to BC and AM perp. to BC or BC produced ; so that DM is the projection of AD on BC.

To prove that $AB^2 - AC^2 = 2BC.DM$.

$$\begin{aligned} \text{Now } AB^2 - AC^2 &= (BM^2 + AM^2) - (CM^2 + AM^2) \\ &= BM^2 - CM^2 \\ &= (BM + CM)(BM - CM) \\ &= \left.\begin{array}{l} BC.2DM \ [\text{ Fig. } 1 \ ] \\ \text{or } 2DM.BC \ [\text{ Fig. } 2 \ ] \end{array}\right\} \qquad [\textit{App.} \text{ IV.} \\ &= 2BC.DM. \end{aligned}$$

COR. *In any triangle, if a perpendicular is drawn from the vertex to the base the difference of the squares on the sides is equal to the difference of the squares on the segments of the base.*

That is, $AB^2 - AC^2 = BM^2 - CM^2$

Ex. 4. A point moves so that the difference of the squares on its distances from two fixed points is constant; prove that the locus of the point is a perpendicular to the join of the fixed points.

[Let B, C be the fixed pts. and A the moving pt., so that $AB^2 - AC^2$ = const. (See Figs. of IV. 10.)

Now $AB^2 - AC^2 = 2BC.DM =$ const. But BC is constant ; ∴ DM is constant, *i.e.*, M is a fixed point. Hence the projection of A on BC is a fixed point and therefore the reqd. locus is the perp. to BC at M.]

Ex. 5. BC is the base of an isosceles triangle ABC ; PQ is any straight line parallel to BC, cutting AB, AC in P and Q ; prove that $BQ^2 - CQ^2 = BC.PQ$.

Ex. 6. ABC is a triangle; BM, CN are perpendiculars to the external bisector of the angle A ; prove that the squares on the sum of AB, AC is greater than the square on BC by four times the rect. BM,CN.

[Produce BA, CN to meet in D. Then AD=AC, and CN=DN ; also BD=AB+AC. Draw BP perp. to CD. Then BPNM is a rect., and **BM**$=$**PN**. Thus **BN** is a median of the $\triangle$**BCD** and **NP** its projection on **CD**. Hence $BD^2-BC^2=2CD.PN=4CN.BM$].

Ex. 7. **ABC** is a triangle; **BM, CN** are perpendiculars to the internal bisector of the angle **A**; prove that the square on the difference of **AB, AC** is less than the square on **BC** by four times the rect. **BM,CN.**

[Let $AB>AC$. Produce **CN** to cut **AB** in **D**. Then $AD=AC$, and $CN=DN$; also $BD=AB-AC$. Draw **BP** perp. to **CD**. Then **BPNM** is a rect.; so that $BM=PN$. Now **BN** is a median of the $\triangle$**BCD**, and **NP** its projection on **CD**. Thus $BC^2-BD^2=2CD.PN=4CN.BM$.]

5. THEOREM. *If a straight line is divided into any two parts, then* (i) *the rectangle contained by the two parts is a maximum and* (ii) *the sum of their squares is a minimum, when the two parts are equal.*

Let AB be divided at X into any two parts.

To prove that

(i) AX.XB is greatest, when X is the mid point of AB;

(ii) AX^2+BX^2 is least, when X is the mid point of AB.

Proof. Let C be the mid point of AB.

By IV. 7, $AX.XB=AC^2-CX^2$.

(i) Now AC^2 being constant, the rect. AX,XB increases as CX^2 (and therefore CX), diminishes; and becomes greatest when CX is least, or 0; that is, when X coincides with C.

Thus AX.XB is a maximum when X is the mid point of AB; that is, when $AX=XB$.

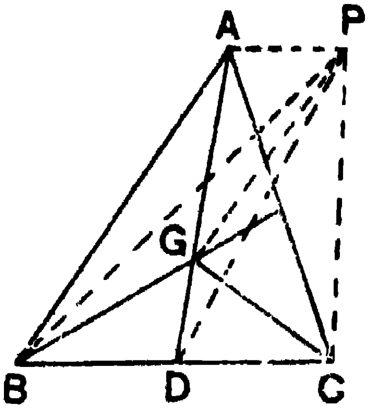
(ii) Again $AX^2+BX^2=2AC^2+2CX^2$ [IV. 6.

Now AC^2 being constant, AX^2+BX^2 is least when CX^2 is so, *i.e.*, when $CX=0$, or when X coincides with C.

Thus AX^2+BX^2 is a minimum, when X is the mid point of AB; that is, when $AX=XB$.

6. THEOREM. *If* G *is the centroid of a triangle* ABC *and* P *any point, prove that*

$$PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3GP^2$$



Because G is a point of trisection of AD.

∴ by (LXV) 14,

$2PD^2+PA^2=6GD^2+3GP^2$.

By *Apoll.*,

$PB^2+PC^2=2BD^2+2PD^2$.

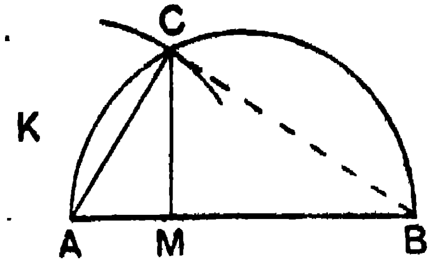
Hence $PA^2+PB^2+PC^2$

$=6GD^2+2BD^2+3GP^2$

$=4GD^2+2GD^2+2BD^2+3GP^2$.

$=GA^2+GB^2+GC^2+3GP^2$ (∵ $GA=2GD$.)

7. PROBLEM. *To divide a given straight line into two parts, so that the rectangle contained by the whole line and one part may be equal to the square on a given straight line.*



Let AB and K be two given straight lines.

To divide AB *at* M, *so that* $AM.AB=K^2$.

Construction. On AB, describe a semicircle. With centre A and radius K, draw an arc to cut the semicircle at C. Draw CM perpendicular to AM.

Then AB is divided at M as required.

Proof. Join BC; then the ∠ ACB is a right angle [(XIV), 21].

Now CM is drawn perp. to AB from the right angle ACB; therefore $AM.AB=AC^2=K^2$ [IV. 5, *Cor.* 1].

MISCELLANEOUS EXERCISES.

1. **ABC** is a triangle and **X** a point in **BC** such that $AB^2+BX^2=AC^2+CX^2$; if **M** is the middle point of **AX**., prove that $BM=CM$.

2. **G** is the centroid of an equilateral triangle **ABC**; prove that

$$AB^2=3AG^2.$$

3. **AB** is a straight line and **PM** is perpendicular to **AB**, or **AB** produced; **Q** is any point in **PM**, or **PM** produced, shew that

$$PA^2+QB^2=PB^2+QA^2.$$

4. If from the middle point of one of the sides of a right-angled triangle a perpendicular is drawn to the hypotenuse, the difference of the squares on the segments so formed is equal to the square on the other side.

5. If a straight line is divibed into three parts, the square on the whole line is equal to the sum of the squares on each part together with twice the rectangle contained by every two of them.

6. **In a right-angled triangle if a perpendicular is drawn from the right angle to the hypotenuse, the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse.**

7. **ABC** is a triangle and **AD** is perpendicular to **BC**; if $AD^2=BD.DC$, prove that the $\angle BAC$ is a right angle.

8. **In a right angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on either side containing the right angle is equal to the rectangle contained by the hypotenuse and that segment of it which is adjacent to that side.**

9. **ABC** is a triangle and **AD** is a perpendicular to **BC**; if $AB^2=BC.BD$, shew that the $\angle BAC$ is a right angle.

10. The altitudes **AP, BQ, CR** of a triangle intersect at **O**; shew that

(i) $AO.OP=BO.OQ=CO.OR$;

(ii) $AP.PO=BP.PC$, $BQ.QO=CQ.QA$, $CR.RO=AR.RB$.

11. Make a rectangle equal to the difference of two given squares.

12. A and B are two fixed points; find the locus of the point P which moves so that the difference of the squares on AP, BP is equal to the square on AB.

13. Any rectangle is half the rectangle contained by the diagonals of the squares upon its adjacent sides.

14. Divide a given straight line into two parts, so that the sum of the squares on them may be equal to a given square. When is this problem impossible?

15. Two chords AB, CD of a circle of centre O cut at right angles at X; shew that $AB^2+CD^2+4OX^2$ is equal to twice the square on a diameter.

16. ABC is an acute-angled triangle; from A, B, C, perpendiculars AD, BE, CF are drawn to BC, CA and AB respectively; prove that $BC^2+CA^2+AB^2=2(AB.AF+BC.BD+CA.CE)$.

17. X is a fixed point in a given line AB; find the locus of a point P which moves so that $AP^2-BP^2=AX^2-BX^2$.

18. If the straight line AB is divided internally in medial section at M, then $AB^2+BM^2=3\,AM^2$.

19. $ABCD$ is a rhombus and P any point in AC; shew that the difference of the squares on AB, BP is equal to the rect. AP,PC.

20. In any right-angled triangle, four times the sum of the squares on the medians which bisect the sides containing the right-angle is equal to five times the square on the hypotenuse.

21. Construct a rhombus equal in area to a given parallelogram.

22. Construct a triangle equal to a given triangle and having two of its sides of given lengths. When is this problem impossible?

23. From O, a point of intersection of two circles, lines OAB, OPQ are drawn at right angles to cut the circles, so that A, Q are on one circle and B, P on the other; shew that four times the square on the join of the centres $=AB^2+PQ^2$.

24. How many triangles can be formed by taking three lines out of the six whose lengths are 3, 4, 5, 11, 12 and 13 units? How many of these triangles are (1) right-angled, (2) obtuse-angled, and (3) acute-angled?

25. $ABCD$ is a rectange; E is any point in CD; EC is bisected at F; prove that $AC^2-AE^2=4\,CF.DF$.

26. **AP** is a diameter of a circle circumscribing an equilateral triangle **ABC**; if **O** is the centre of the circle, shew that **OCP** is an equilateral triangle and that $AC^2 = 3OA^2$.

27. Construct a rectangle equal to a given square, having given the sum of two adjacent sides of the rectangle.

28. Produce a side of a triangle, so that the rectangle contained by it and the part produced may be equal to the difference of the squares on the other two sides.

29. From the hypotenuse of a right-angled triangle, portions are cut off equal to the adjacent sides; prove that the square on the middle segment is equal to twice the rectangle contained by the extreme segments.

30. The chord **AB** of a circle is produced both ways, so that **AC**= **BD**; tangents **CE**, **DF** are drawn on opposite sides of **CD**, shew that **EF** bisects **AB**.

31. Find a point **P** in a straight line **AB** produced, such that

$$AB^2 + AP^2 = 2\,AP.PB$$

32. If a parallelogram has one of its angles equal to two-thirde of a right-angle, prove that the difference of the squares on the diagonals is equal to twice the rectangle contained by any two adjacent sides.

33. At **B** in **AB** two equal straight lines **BC**, **BD** are drawn making equal angles with **AB** and **AB** produced; shew that

$$AC^2 \sim AD^2 = 2AB\ CD.$$

34. In any quadrilateral, the sum of the squares on the two sides subtending an obtuse angle at the intersection of the diagonals is greater than the sum of the squares on the two sides subtending an acute angle at that point.

How would this relation be changed, if each pair of sides subtends a right angle?

35. **ABC** is a triangle obtuse-angled at **C**; **AD** is perpendicular to **BC** produced; from **AD** produced **DF** is cut off equal to **AB** and **DG** equal to **AC**, shew that $FC = GB$.

36. If **O** is the point of intersection of the altitudes of an acute-angled triangle **ABC**, prove that $OA^2 + OB^2 + OC^2$ is less than half the sum of the squares on the three sides of the triangle.

37. From a fixed point A outside a given circle of centre O, any two lines APQ, ARS are drawn equally inclined to the diameter through A and cutting the circle at P, Q and R, S; shew that PS and QR intersect at a fixed point on the diameter.

38. The common chord AB of two intersecting circles is produced to cut their common tangents in X and Y; if PT be a common tangent, prove that $XY^2 = PT^2 + AB^2$.

39. Construct a triangle having given its area, base and the sum of the squares on its sides.

40. **In an isosceles right-angled triangle, twice the square on the join of the right angle to any point in the hypotenuse is equal to the sum of the squares on the segments of the hypotenuse.**

41. P is a point in the diaganal AC of a square ABCD; shew that the triangle whose sides are AP, PC and the diagonal of the square drawn on PB, is right-angled.

42. ABC is a triangle and BD is the median bisecting AC; AE is drawn perpendicular to BC; shew that $BD^2 = AD^2 \pm BC.BE$, according as E is in BC or in BC produced.

43. ABC is an acute-angled triangle; circles described on BC and AC as diameters cut the altitudes from A and B in P, Q and R, S, respectively; prove that P, Q, R, S are concyclic.

44. ABC is an acute-angled triangle; AD, BE are perpendiculars to BC and CA intersecting at O; prove that

the rect. AD,OD = the rect. BD,DC.

45. **ABC is an isosceles triangle whose vertex is A. If CX is perpendicular to AB and XP to BC, prove that**

$$AB^2 = PA^2 + PX^2.$$

46. A variable circle passes through two fixed points A and B and cuts a fixed circle in X and Y; if AX, AY cut the the fixed circle again in P and Q, shew that PQ passes through a fixed point.

[*Hint.* Let PQ cut AB at Z. Prove P, X, B, Z are concyclic.]

47. ABC is a triangle; find a point P such that $PA^2 + PB^2 + PC^2$; may be a minimum.

48. P is a point within a triangle ABC such that if perpendiculars PM, PN are drawn to AB, AC, the rect. AB,BM = the rect. AC,CN; shew that P lies on a fixed straight line perpendicular to BC.

49. If two opposite sides of any quadrilateral are bisected, the sum of the squares on the other two sides together with the squares on the diagonals is equal to the sum of the squares on the sides bisected together with four times the square on the join of the points of bisection.

[*Hint.* Join the pt. of bisection of one side to the extremities of the opp. side.]

50. On the sides of any triangle, squares are described all external to the triangle and their adjacent corners are joined; prove that the sum of the squares on these joins is equal to three times the sum of the squares on the sides of the triangle.

Hence shew that if the triangle is right-angled, the sum of the squares on the aforesaid joins is equal to six times the square on the hypotenuse.

51. Divide a given straight line into two parts so that the square on one part may be double the rectangle contained by the whole line and the other part. [See IV. Prob. 3.]

52. Divide a given straight line into two parts, so that the rectangle contained by one part and a given straight line may be equal to the square on the other part.

53. Produce a given straight line so that the sum of the squares on the whole line thus produced and on the part produced may be equal to three times the square on the given line.

54. Prove that, if through a fixed point **O**, any straight line be drawn meeting a fixed circle whose centre is **C** in **P** and **P′**, the circle described round **PCP′** will pass through a fixed point **D** in **OC**; and that if **OC** meet the fixed circle at **A**, **AP** will bisect the ∠**OPD**.

BOOK V

RATIO AND PROPORTION—SIMILAR FIGURES

BOOK V

RATIO AND PROPORTION—SIMILAR FIGURES

INTRODUCTION

Magnitudes are either commensurable or incommensurable. **Commensurable** magnitudes are those which have a *common measure*, *i.e.*, each of which can be expressed as a multiple of a smaller magnitude of the same kind, called the **unit**. Thus 5 ft. and 8 ft. are commensurable magnitudes, for they are respectively 5 times and 8 times the unit, *one foot*. **Incommensurable** magnitudes are those which have no common measure. Thus 2 and $\sqrt{3}$ are incommensurable, for there is no number of which both 2 and $\sqrt{3}$ are multiples.

In modern elementary geometry, the magnitudes treated are assumed to be commensurable, and the proofs given are based on this assumption. Hence though theoretically the treatment of magnitudes in modern geometry is not so satisfactory as that in Euclid, yet the conclusions arrived at are sufficiently accurate for all practical purposes. For in practical life, all calculations are based on measurements, and measurements being expressed in numbers are commensurable.

DEFINITIONS

The **ratio** of one magnitude to another of the same kind is the quotient when the numerical measure of the first is divided by that of the second, the same unit measuring both magnitudes.

The ratio between two magnitudes, one of which contains a units and the other b units, is written $\frac{a}{b}$, or $a : b$, or a/b, or $a + b$.

In a ratio $a : b$, a is called the **antecedent**, b the **consequent**, and both a and b, the **terms** of the ratio.

Note. In forming a ratio, care must be taken to *express each magnitude in terms of the same unit.* Thus the ratio of 3 feet to 15 inches is, not 3 : 15, but 36 : 15.

Four magnitudes are **in proportion**, or are **proportionals**, when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus, if $a : b = m : n$, then a, b, m and n are said to be proportionals.

The proportion "$a : b = m : n$" is read "a is to b as m is to n."

In a proportion $a : b = m : n$, a and n are called the **extremes**, b and m are called the **means**, and n is called a **fourth proportional** to a, b and m.

Three magnitudes of the same kind are proportionals, when the ratio of the first to the **second** is equal to the ratio of the **second** to the third.

Thus a, b, c are proportionals, if $a : b = b : c$; and b is called a **mean proportional** to a and c, and c a **third proportional** to a and b.

In the proof of the Theorems in this Book, we shall have occasion to refer to some theorems, which can be easily shewn to be true for commensurable quantities by the methods of Algebra.

Lemma 1. Equal magnitudes have the same ratio to the same magnitude; and *conversely*, the same magnitude has the same ratio to equal magnitudes.

That is, (i) if $a = b$, then $\frac{a}{x} = \frac{b}{x}$; (ii) if $\frac{x}{a} = \frac{x}{b}$, then $a = b$.

Lemma 2. Ratios that are equal to the same ratio are equal to one another.

That is, if $\frac{a}{b} = \frac{x}{y}$ and $\frac{m}{n} = \frac{x}{y}$, then $\frac{a}{b} = \frac{m}{n}$.

Lemma 3. Equimultiples of two magnitudes have the same ratio as the magnitudes themselves.

$$\text{For } \frac{ma}{mb} = \frac{a}{b}.$$

Lemma 4. If four magnitudes are proportionals, they are proportionals when taken inversely. [*Invertendo.*]

That is, if $\frac{a}{b} = \frac{m}{n}$, then $\frac{n}{m}$

Lemma 5. If four magnitudes of the same kind are proportionals, they are proportionals when taken alternately. [*Alternando.*]

That is, if $\frac{a}{b} = \frac{m}{n}$, then $\frac{a}{m} = \frac{b}{n}$.

Lemma 6. If $\frac{a}{b} = \frac{x}{y}$, and $\frac{b}{c} = \frac{y}{z}$, then $\frac{a}{c} = \frac{x}{z}$.

[*Ex. Æquali.*]

Lemma 7. If three straight lines are proportionals, the *first* has the same ratio to the *third* that the square on the first has to the square on the second, or that the square on the second has to the square on the third.

That is, if $\frac{a}{b} = \frac{b}{c}$, then $\frac{a}{c} = \frac{a^2}{b^2}$ or $= \frac{b^2}{c^2}$.

For $\frac{b}{c} = \frac{a}{b}$ and $\frac{a}{b} = \frac{a}{b}$;

∴ compounding, $\frac{b}{c} \times \frac{a}{b} = \frac{a^2}{b^2}$ or $- = -$

Lemma 8. If four numbers are proportionals, the product of the extremes is equal to the product of the means.

That is, if $\frac{a}{b} = \frac{m}{n}$, then $an = bm$.

Hence if four straight lines are proportionals, *the* **rectangle** *contained by the extremes* = *the* **rectangle** *contained by the means.*

Lemma 9. If $\frac{a}{b} = \frac{m}{n}$, then $\frac{a+b}{b} = \frac{m+n}{n}$.

[*Componendo.*]

Lemma 10. If $\frac{a}{b}=\frac{m}{n}$, then $\frac{a-b}{b}=\frac{m-n}{n}$.

[*Dividendo.*]

Lemma 11. If $\frac{a}{b}=\frac{m}{n}$, then $\frac{a+b}{a-b}=\frac{m+n}{m-n}$.

[*Componendo and Dividendo.*]

This Lemma follows from Lemmas 9 and 10.

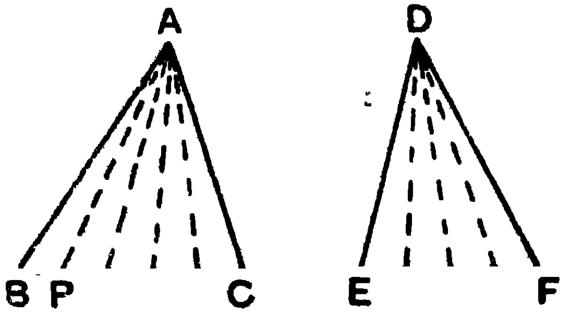
Lemma 12. If $\frac{a}{b}=\frac{m}{n}$.

then $\frac{a}{a+b}=\frac{m}{m+n}$ or $\frac{a}{a-b}=\frac{m}{m-n}$.

Lemma 13. If $\frac{a}{m}=\frac{b}{n}=\frac{c}{p}=$ ...., then each of these ratios $=\frac{a+b+c+\ldots\ldots\ldots}{m+n+p+\ldots\ldots}$, [*Addendo.*]

*THEOREM 1. [Euc. VI. 1.]

Triangles of equal altitude have the same ratio as their bases.



Let ABC, DEF be two triangles of equal altitude whose bases are BC and EF.

To prove that $\triangle ABC : \triangle DEF = BC : EF$.

Proof. Let BC and EF have a common measure m; so that both BC and EF can be divided into equal parts each of which is m.

Let BC be divided into p and EF into q equal parts.

Then $BC = pm$ and $EF = qm$;

$$\therefore BC : EF = pm : qm = p : q.$$

Again let BP be one of the equal parts of BC.

Join A to the points of division of BC, and D to those of EF.

Then all the triangles into which the $\triangle^s$ ABC, DEF are divided are on equal bases and have the same altitude;

$\therefore$ they are equal to one another and each is equal to the $\triangle ABP$.

$\therefore$ the $\triangle ABC = p \times \triangle ABP$ and the $\triangle DEF = q \times \triangle ABP$;

$$\therefore \triangle ABC : \triangle DEF = p \times \triangle ABP : q \times \triangle ABP = p : q.$$

Hence $\triangle ABC : \triangle DEF = BC : EF$. Q. E. D.

Alternative Proof of Theorem 1.

Let h denote the altitude of the $\triangle^s$ ABC and DEF.

Then the $\triangle ABC = \frac{1}{2} BC.h$, and the $\triangle DEF = \frac{1}{2} EF.h$.

Hence $$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}\ BC.h}{\frac{1}{2}\ EF.h} = \frac{BC}{EF};$$

That is, $$\triangle ABC : \triangle DEF = BC : EF.$$

Q. E. D.

Cor. 1. *Parallelograms of equal altitude have the same ratio as their bases.*

Cor. 2. *Triangles, or parallelograms, on the same base or on equal bases, have the same ratio as their altitudes.*

THEOREM 2. [Euc. VI. 2.]

(i) If a straight line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.

(ii) Conversely, if a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

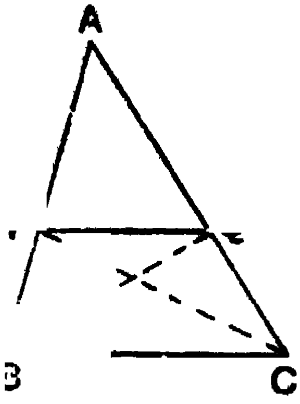


Fig. 1.

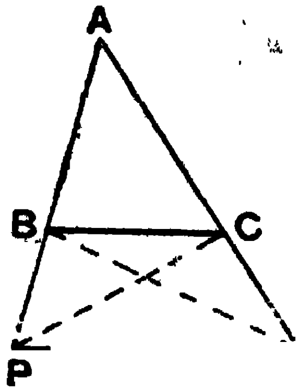


Fig. 2.

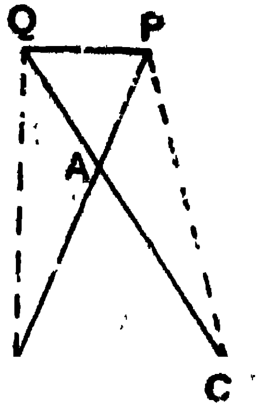


Fig. 3.

Let ABC be a triangle.

(i) Let PQ, drawn parallel to BC, cut AB, AC, or these sides produced, at P and Q.

To prove that AP : PB = AQ : QC.

Construction. Join BQ, CP.

Proof. Because PQ is parl. to BC;

$\therefore$ the $\triangle BPQ =$ the $\triangle CPQ$; [II. 3.

$\therefore$ $\triangle APQ : \triangle BPQ = \triangle APQ : \triangle CPQ$. [*Lem.* 1.

Now triangles of equal altitude have the same ratio as their bases;

$\therefore$ $\triangle APQ : \triangle BPQ = AP : PB$,

and $\triangle APQ : \triangle CPQ = AQ : QC$. [V. 1.

But the $\triangle BPQ =$ the $\triangle CPQ$; [*Proved.*

Hence $AP : PB = AQ : QC$.

(ii) *Conversely*, let PQ divide AB and AC or these sides produced, proportionally ; so that $AP : PB = AQ : QC$.

To prove that PQ *is parallel to* BC.

Construction. Join BQ, CP.

Proof. Because the Δ^s APQ and BPQ are of the same altitude,

$\therefore \; \Delta APQ : \Delta BPQ = AP : PB.$ [V. 1.

Similarly $\Delta APQ : \Delta CPQ = AQ : QC.$

$\therefore \; \Delta APQ : \Delta BPQ = \Delta APQ : \Delta CPQ;$

$\therefore$ the Δ BPQ = the Δ CPQ. [*Lem.* 1.

But these triangles are on the same base PQ and on the same side of it ;

$\therefore$ PQ is parallel to BC. [II. 4, *Cor.* 1.

Q. E. D.

Cor. *If* PQ *is parl. to* BC, *then* $AP : AB = AQ : AC$; and conversely *if* $AP : AB = AQ : AC$, *then* PQ *is parallel to* BC.

(i) Fig. 1, $\dfrac{AP}{PB} = \dfrac{AQ}{QC}, \; \therefore \; \dfrac{AP}{AP+PB} = \dfrac{AQ}{AQ+QC}$, *i.e.*, $\dfrac{AP}{AB} = \dfrac{AQ}{AC}$.

(ii) Fig. 2, $\dfrac{AP}{PB} = \dfrac{AQ}{QC}, \; \therefore \; \dfrac{AP}{AP-PB} = \dfrac{AQ}{AQ-QC}$, *i.e.*, $\dfrac{AP}{AB} = \dfrac{AQ}{AC}$.

(iii) Fig. 3, $\dfrac{AP}{PB} = \dfrac{AQ}{QC}, \; \therefore \; \dfrac{AP}{PB-AP} = \dfrac{AQ}{QC-AQ}$, *i.e.*, $\dfrac{AP}{AB} = \dfrac{AQ}{AC}$.

Similarly the *convers* can be proved.

Alternative Proof of Theorem 2.

THEOREM 2. [Euc. VI. 2.]

(i) **If a straight line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.**

(ii) **Conversely, if a straight line divides two sides of a triangle proportionally, it is parallel to the third side.**

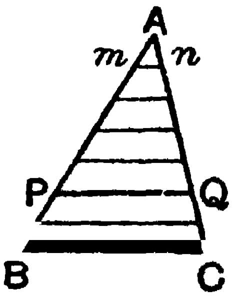


Fig. 1.

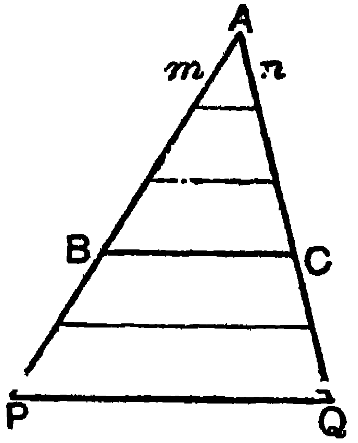


Fig. 2.

Let ABC be a triangle.

(i) Let a st. line PQ, drawn parallel to BC, cut AB, AC, or these sides produced, at P and Q.

To prove that AP : PB = AQ : QC.

Proof. Let AP, PB have a common measure m; then both AP, PB can be divided into equal parts, each of which is m.

Let AP be divided into p and PB into q such parts; so that AP $= pm$ and PB $= qm$;

$$\therefore \text{AP} : \text{PB} = pm : qm = p : q.$$

Through the points of division in AP, PB, draw parallels to BC. Then these parallels divide AQ, QC into p and q equal parts. [I. 26.

Let each of these parts be n; so that $AQ = pn$ and $QC = qn$;

$$\therefore AQ : QC = pn : qn = p : q.$$

Hence $\quad AP : PB = AQ : QC.$

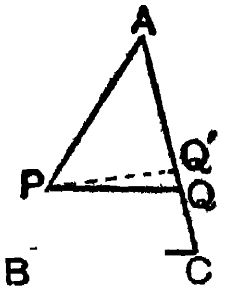


Fig. 1.

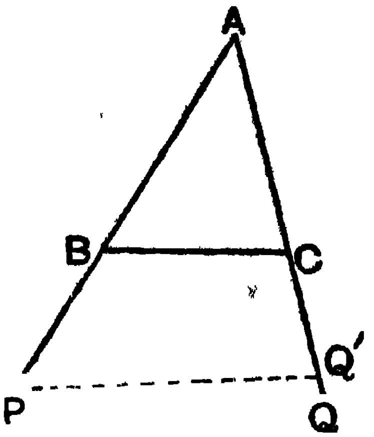


Fig. 2.

(ii) *Conversely*, let a st. line PQ divide AB, AC, or these sides produced, proportionally,

so that $AP : PB = AQ : QC$.

To prove that PQ *is parallel to* BC.

Proof. If PQ is not parallel to BC, let PQ′ be drawn through P, parallel to BC to meet AC in Q′

Then $AP : PB = AQ' : Q'C$.

But $AP : PB = AQ : QC$. [*Hyp.*

$\therefore AQ' : Q'C = AQ : QC$;

$\therefore$ by *Lem.* 12, $AQ' : AQ' + Q'C = AQ : AQ + QC$ [fig. 1].

or $AQ' : AQ' - Q'C = AQ : AQ - QC$ [fig. 2];

that is, $AQ' : AC = AQ : AC$ [in both cases];

$\therefore AQ' = AQ$.

Thus the points Q′ and Q coincide,

that is, PQ′ coincides with PQ;

$\therefore$ PQ is parallel to BC. Q. E. D.

EXERCISES LXXIV.

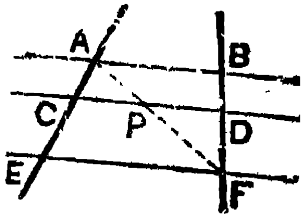
1. **Three parallels cut any two transversals proportionally.**

Let the parls. **AB, CD, EF** cut two transversals, in **A, C, E** and **B, D, F.**

To prove that $\frac{AC}{CE} = \frac{BD}{DF}$.

Join **AF,** cutting **CD** in **P.**

Then $\frac{AC}{CE} = \frac{AP}{PF} = \frac{BD}{DF}$.



2. **A series of parallels cuts any two transversals proportionally.**

3. The line joining the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

4. **OAP, OBQ, OCR** are three straight lines ; points **A** and **P, B** and **Q, C** and **R,** on them are such that **AB** is parallel to **PQ** and **BC** to **QR.** Shew that **AC** is parallel to **PR.**

5. Through any point **D** in the side **AB** of a triangle **ABC, DE** is drawn parallel to **BC** cutting **AC** in **E** ; **CF** is parallel to **EB** meeting **AB** produced in **F** ; prove that **AD : AB=AB : AF.**

6. **ABC, ABD** are two triangles ; **M, N** are the middle points of **BC** and **BD** respectively ; prove that **MN** is parallel to the join of the centroids of the triangles.

7. Triangles **APB, AQB,** are on a common base **AB,** and from any point **C** in **AB,** parallels are drawn to **AP** and **AQ** respectively meeting **BP** in **X** and **BQ** in **Y.** Prove that **XY** is parallel to **PQ.**

8. **OPQ** is a straight line drawn through a fixed point **O** and is such that **OP : OQ** is constant. If **P** moves along a fixed straight line, find the locus of **Q.**

9. **D** is any point in **BC,** the base of a triangle **ABC ; DP, DQ** are drawn parallel to **AC, AB** respectively, meeting **AB** in **P** and **AC** in **Q.** Find the locus of the middle point of **PQ.**

10. Straight lines **AOB, COD** intersect at **O,** and **AO : OB =CO : OD** ; if **P, Q** are the middle points of **AB, CD,** prove that **PQ** is parallel to **AC** and **BD.**

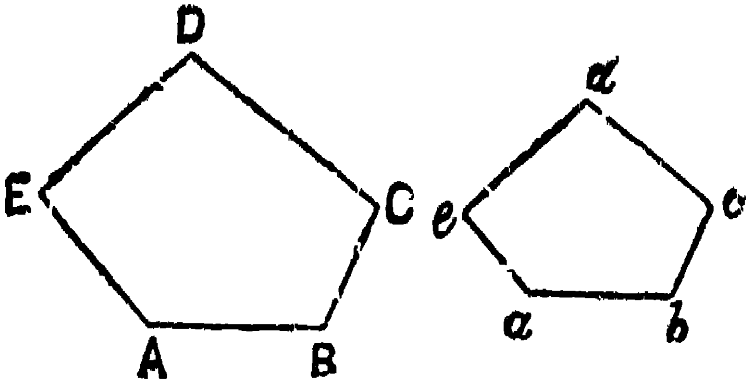
11. If two triangles are on equal bases and between the same parallels, the intercepts made by the sides of the triangles on any parallel to the line of bases are equal [*App.* II. 8.]

Hence shew that the parallels cuts off equal areas from the triangles.

DEFINITIONS

1. Two triangles, or two polygons, are said to be **equiangular** to one another, if the angles of the one are equal to the angles of the other, when taken in the same order.

2. Two triangles, or two polygons, are said to be **similar**, if their angles when taken in order are equal and the sides about each pair of equal angles are proportional.



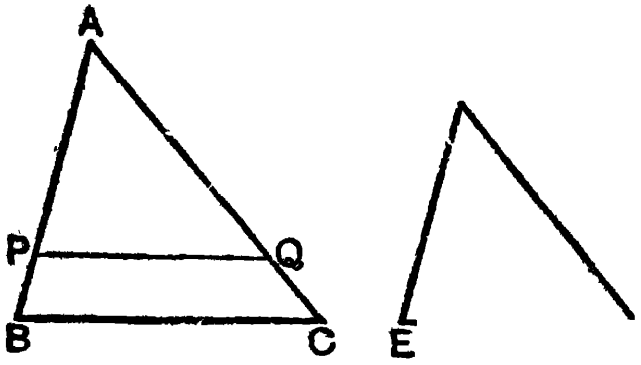
Thus ABCDE and *abcde* are two similar polygons, if $\angle A = \angle a$, $\angle B = \angle b$, $\angle C = \angle c$, $\angle D = \angle d$ and $\angle E = \angle e$, and $AB : ab = BC : bc = CD : cd = DE : de = EA : ea$.

3. The sides about the equal angles of similar polygons are called **corresponding sides**; and the vertices of equal angles are called **corresponding vertices.**

Thus in the above figures, the *corresponding sides* are AB and *ab*, BC and *bc*, &c.; and the *corresponding vertices* are A and *a*, B and *b*, &c.

THEOREM 3. [Euc. VI. 4.]

If two triangles are equiangular, their corresponding sides are proportional.



Let ABC, DEF be two triangles, in which the $\angle A =$ the $\angle D$, the $\angle B =$ the $\angle E$ and the $\angle C =$ the $\angle F$.

To prove that $AB : DE = AC : DF = BC : EF$.

Construction. From AB, AC cut off AP, AQ equal to DE and DF respectively. Join PQ.

Proof. In the $\triangle^s$ APQ, DEF,

$$\left.\begin{array}{ll}\text{(i)} & AP = DE, \\ \text{(ii)} & AQ = DF,\end{array}\right\} \qquad [\textit{Const.}$$

and (iii) the $\angle A =$ the $\angle D$; [*Hyp.*

$\therefore$ the triangles are congruent.

$\therefore$ the $\angle APQ =$ the $\angle E =$ the $\angle B$; [*Hyp.*

$\therefore$ PQ is $\parallel$ to BC.

$\therefore AB : AP = AC : AQ$; [V. 2, *Cor.*

that is, $AB : DE = AC : DF$.

Similarly by cutting off lengths from BA and BC equal to ED and EF respectively, it can be proved that

$$AB : DE = BC : EF.$$

$$\therefore AB : DE = AC : DF = BC : EF.$$

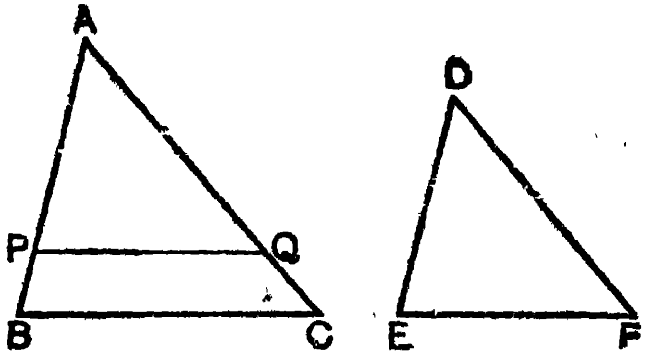
Q. E. D.

Cor. *If two triangles are equiangular, they are similar.*

THEOREM 4. [Euc. VI. 5.]

[Converse of Theorem 3.]

If the sides of two triangles when taken in order are proportional, the triangles are equiangular, those angles being equal which are opposite to corresponding sides.



Let ABC, DEF be two triangles, in which

AB : DE = AC : DF = BC : EF.

To prove that *the* △ˢ ABC, DEF *are equiangular.*

Construction. From AB, AC, cut off AP, AQ equal to DE, DF respectively. Join PQ.

Proof. Because AB : DE = AC : DF, [*Hyp.*

and DE = AP, DF = AQ;

∴ AB : AP = AC : AQ; [*Const.*

∴ PQ is parallel to BC;

∴ the ∠ APQ = the corresponding ∠ B,

and the ∠ AQP = the corresponding ∠ C.

Hence the △ˢ ABC, APQ are equiangular;

∴ AB : AP = BC : PQ; [V. 3.

that is, AB : DE = BC : PQ.

But AB : DE = BC : EF,

∴ BC : PQ = BC : EF;

∴ PQ = EF.

And AP = DE, and AQ = DF. [*Const.*

Thus the Δ^s **APQ, DEF** are congruent ;

so that the $\angle$ **APQ**=the $\angle$ **E**, and the $\angle$ **AQP**=the $\angle$ **F**.

But the $\angle$ **APQ**=the $\angle$ **B**, and the $\angle$ **AQP**=the $\angle$ **C** ; [*Proved.*

$\therefore$ the $\angle$ **B**=the $\angle$ **E** and the $\angle$ **C**=the $\angle$ **F** ;

consequently the $\angle$ **A**=the $\angle$ **D**.

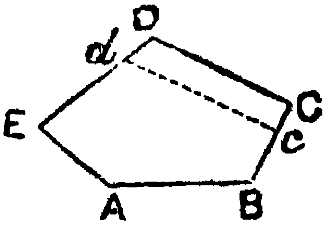
$\therefore$ the Δ^s **ABC, DEF** are equiangular.

Q. E. D.

Cor. *If two triangles have their sides proportional when taken in order, they are similar.*

Note. From Theorem 3, it is seen that two equiangular *triangles* are *similar*. But two equiangular *polygons* are not *necessarily similar*.

Thus if *cd* be drawn parl. to **CD**, then it is evident that **ABCDE** and **AB***cd***E** are equiangular but their *corresponding* sides are not proportional ; for then **AB** : **BC** would be equal to **AB** : **B***c*. *i.e.*, **BC** would be equal to **B***c*, which is absurd.



Hence **two conditions** are necessary for the similarity of two **polygons**, *viz.*, the equality of the angles **and** the proportionality of the corresponding sides ; but only **one condition** is sufficient for the similarity of **two triangles**, *viz.*, the equality of the angles [V. 3], or the proportionality of the sides [V. 4].

EXERCISES LXXV.

1. **Any parallel to a side of a triangle meeting the other two sides, cuts off a similar triangle.**

2. The line joining the middle points of any two sides of a triangle is parallel to the third side and half of it.

3. **The diagonals of a trapezium cut one another in the same ratio.**

4. Any two medians of a triangle cut one another at a point of trisection.

Hence prove that the medians of a triangle are concurrent.

5. **ABC** is a triangle and **AD** is any line from **A** to the base **BC**. If any parallel to **BC** cuts **AB**, **AD**, **AC** in **P**, **Q**, **R**, shew that

$$PQ : QR = BD : DC.$$

Hence **any parallel to a side of a triangle is bisected by the corresponding median.** [See *App.* II. (7).]

6. **ABC, DEF are two equiangular triangles. If AM makes the same angle with BC as DN makes with EF, prove that**

$$AM : DN = BC : EF.$$

Hence evidently when **AM**, **DN** are perpendiculars to **BC**, **EF**,

$$AM : DN = BC : EF.$$

7. The line joining the middle points of the parallel sides of a trapezium passes through the intersection of the diagonals.

8. **The line drawn parallel to the parallel sides of a trapezium through the point of intersection of the diagonals is bisected at the point.**

9. **The point of intersection of the oblique sides of a trapezium, the middle points of the parallel sides and the intersection of the diagonals are collinear.**

10. **AB**, **CD** are chords of a circle intersecting at **P** inside or outside the circle; prove that the triangles **ACP**, **DBP** are similar.

Hence shew that the rect. **AP**, **PB** = the rect. **CP**, **PD**. [IV. 11.]

11. From a point **P** outside a circle, a tangent **PT** and a secant **PAB** are drawn, the latter to cut the circle at **A** and **B**. Prove that

$$PA : PT = PT : PB \; ; \textit{i.e.}, \; PT^2 = PA.\,AB \quad \text{[IV. 11, Cor.]}$$

12. **Two circles touch one another at A and a straight line through A cuts them at P and Q. Prove that their diameters are as AP : AQ.**

13. Two circles intersect at **A** and **B**, and through **B**, any line **PBQ** is drawn to cut the circles at **P** and **Q**. Prove that the radii of the circles are as **AP** : **AQ**.

Note. Hence the ratio **AP** : **AQ** is constant for all positions of **PBQ**.

14. **A common tangent to two circles cuts the line joining their centres, internally or externally, in the ratio of their radii.**

15. **If from a point P in either arm of an angle XOY, PM is drawn perpendicular to the other arm, prove that each of the ratios**

$$PM : OP, \quad OM : OP \text{ and } PM : OM$$

is constant as long as the angle XOY is constant.

[*Hint.* Take **P** in **OY**. Take another pt. **P′** in **OY** and draw **P′M′** perp. to **OX**. Prove that $\frac{PM}{OP}=\frac{P'M'}{OP'}$. Again take another pt. **P″** in **OX** and draw **P″M″** perp. to **OY**, and prove that $\frac{PM}{OP}=\frac{P''M''}{OP''}$.

Thus the ratio $\frac{PM}{OP}$ is constant.

Similarly prove the other two ratios to be constant.]

Def. The ratio $\frac{PM}{OP}$ is called the **sine** of the $\angle XOY$, $\frac{OM}{OP}$ is called the **cosine** of the $\angle XOY$ and $\frac{PM}{OM}$ is called the **tangent** of the $\angle XOY$.

Hence *the* sine, *the* cosine *and the* tangent *of an angle are constant if the angle is invariable.*

Note. *Sin*, *cos* and *tan* are often used as abbreviations for the words sine, cosine and tangent respectively. Sin **A**, cos **A**, tan **A** are called **trigonometrical ratios** of the angle **A**.

16. **ABC is a triangle right-angled at C. Prove that**
(i) $AC=AB\cos A$, (ii) $BC=AB\sin A$, and (iii) $BC=AC\tan A$.

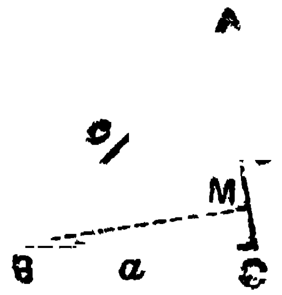
17. **Prove that the area of a triangle is equal to half the product of any two sides into the sine of the included angle.**

That is, the $\triangle ABC=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$.

Let **ABC** be a △ and let a, b, c, be the measures of the sides opposite the angles **A**, **B** and **C** respectively.

Draw **BM** perp. to **AC**.

Then the area of the $\triangle ABC=\frac{1}{2}AC.BM$
$=\frac{1}{2}AC.AB\sin A$ ($\because$ $BM=AB\sin A$)
$=\frac{1}{2}bc\sin A$.



Similarly it may be shewn that $\frac{1}{2}ca\sin B$ and $\frac{1}{2}ab\sin C$ are expressions for the area of the △**ABC**.

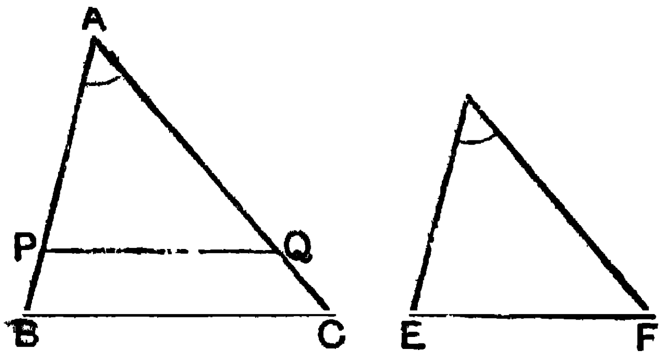
18. Prove that, in a triangle **ABC**,

$$\text{(i)}\quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

(ii) $a=b\cos C+c\cos B$, $b=c\cos A+a\cos C$, $c=a\cos B+b\cos A$.

THEOREM 5. [Euc. VI. 6.]

If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar.



Let ABC, DEF be two triangles, in which

the $\angle A =$ the $\angle D$, and $AB : DE = AC : DF$.

To prove that *the* $\triangle^s$ ABC, DEF *are similar.*

Construction. From AB, AC cut off AP, AQ equal to DE, DF respectively.

Join PQ.

Proof. In the $\triangle$s APQ, DEF,

(i) $AP = DE$, } [*Const.*
(ii) $AQ = DF$, }
and (iii) the $\angle A =$ the $\angle D$; [*Hyp.*

$\therefore$ the $\angle APQ =$ the $\angle E$ and the $\angle AQP =$ the $\angle F$.

Now $AB : DE = AC : DF$; [*Hyp.*

$\therefore AB : AP = AC : AQ$.

$\therefore$ PQ is parallel to BC;

$\therefore$ the $\angle APQ =$ the corresponding $\angle B$,

and the $\angle AQP =$ the corresponding $\angle C$.

But the $\angle APQ =$ the $\angle E$, and the $\angle AQP =$ the $\angle F$. [*Proved.*

$\therefore$ the $\angle B =$ the $\angle E$, and the $\angle C =$ the $\angle F$;

Hence the $\triangle^s$ ABC, DEF are equiangular;

and consequently their corresponding sides are proportional;

$\therefore$ the $\triangle^s$ ABC, DEF are similar. Q. E. D.

*THEOREM 6. [Euc. VI. 7.]

If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, then the third angles are either equal or supplementary; and when equal, the triangles are similar.

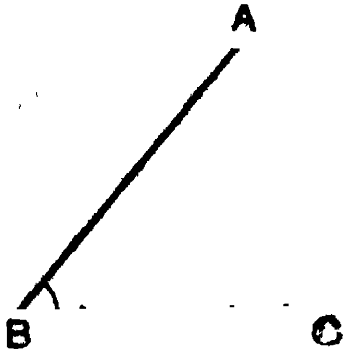


Fig. 1.

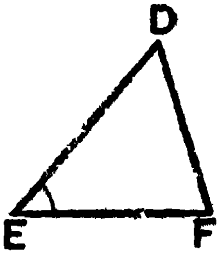


Fig. 2.

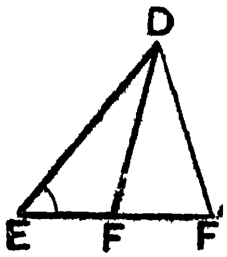


Fig. 3.

Let ABC [Fig. 1.] and DEF [Figs. 2 and 3] be two triangles, such that the ∠B = the ∠E and the sides about the ∠s A and D proportional, so that AB : DE = AC : DF.

To prove that

either (i) *the* ∠C = *the* ∠DFE [Figs. 1 and 2];

or (ii) *the* ∠C = *the supplement of the* ∠DFE [Figs. 1 and 3].

Proof. (i) If the ∠A = the ∠D [Figs. 1 and 2]:

then the ∠C = the ∠F;

thus the Δs ABC, DEF are equiangular and therefore similar. [V. 3, *Cor.*

(ii) But if the ∠A is not equal to the ∠EDF [Figs. 1 and 3], make the ∠EDF′ = the ∠A;

then the ∠C = the ∠F′.

Thus the Δs ABC, DEF′ are equiangular;

∴ AB : DE = AC : DF′.

But AB : DE = AC : DF. [*Hyp.*

∴ AC : DF′ = AC : DF;

∴ DF′ = DF;

∴ the ∠DFF′ = the ∠F′;

∴ the ∠**C** = the ∠**DFF′**.

But the ∠s **DFF′** and **DFE** are supplementary.

∴ the ∠s **C** and **DFE** are supplementary.

Q. E. D.

Cor. *If two triangles have one angle of the one equal to one angle of the other and the sides about one other angle in each proportional, then if the third angles are both acute or both obtuse, the triangles are similar.*

For if the angles **C** and **F** are both acute or both obtuse; then they *cannot be supplementary* and must therefore be *equal.*

Obs. The student should notice the **four conditions** necessary and sufficient for the **similarity** of two triangles. They are stated in (1) V. 3, *Cor.*, (2) V. 4, *Cor.*, (3) V. 5, and (4) V. 6, *Cor.*

EXERCISES LXXVI.

1. **ABCD** is a parallelogram; **AB** is produced to **P** so that **BP** is half of **AB**; if **Q** is the middle point of **BC**, shew that the triangles **ABD**, **BPQ** are similar, and **BD** is double of **PQ**.

2. **ABC**, **DEF** are two similar triangles; **P**, **Q** are points in **BC** and **EF** such that **BP : EQ = BC : EF**. Prove that

   (i) △s **ABP**, **DEQ** are similar; so also are △s **APC**, **DQF**;
   (ii) **AP : DQ = BC : EF**.

3. In equiangular triangles, the medians make equal angles with the corresponding sides.

4. Prove by Theorem 5 that the line joining the middle points of the sides of a triangle is parallel to the base and half of it.

5. Two straight lines **AB**, **CD** intersect internally or externally at **O**, so that **OA : OC = OD : OB**. Prove that the points **A**, **C**, **B**, **D**, are concyclic.

6. **OAB is a straight line; through A, B parallels AX, BY are drawn in the same sense so that OA : OB = AX : BY. Prove that OXY is a straight line.**

7. **ABC** is a triangle right-angled at **A**; a point **D** is taken in the hypotenuse **BC** such that **BC : BA = BA : BD**. Prove that **AD** is perpendicular to **BC**.

8. In a triangle **ABC**, **AD** is drawn perpendicular to **BC**. Prove that if $\mathbf{AB}^2 = \mathbf{BC.BD}$, then the **∠A** is a right angle.

9. **PM**, **QN** are the perpendiculars from points **P** and **Q** to a given straight line **OX**. If **PM : QN = OP : OQ**, prove that **O**, **P**, **Q** are in the same straight line.

[*Hint.* Join **OP**, **OQ**. The **∠**s **POM**, **QON** are acute angles ; ∴ by Theorem 6, *Cor.*, they cannot be supplementary ; hence they are equal ; ∴ **OP** coincides with **OQ**.]

10. **AD**, **BE** are the altitudes of the triangle **ABC** cutting one another at **O**. Prove that **BD : DO = AD : DC**.

11. By the help of Ex. 10, prove that the three altitudes of a triangle are concurrent.

[*Hint.* Join **CO** and prove **∆**s **ADB**, **CDO** similar.]

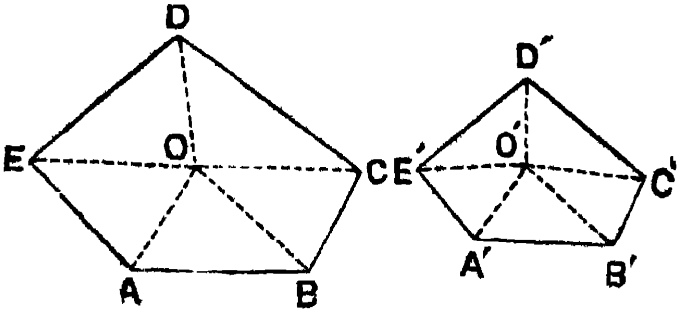
12. **PM**, **QN** are the perpendiculars from **P** and **Q** to a given line **AB** ; **PN**, **QM** intersect at **R** ; **RK** is perpendicular to **AB**. Prove that **PK**, **QK** make equal angles with **AB**.

13 If the ratio of the tangents drawn from a given point **O** to two given circles be equal to the ratio of the radii of the corresponding circles, shew that the two circles will subtend equal angles at **O**.

14, **ABCD** is a quadrilateral in which **CD** touches the circle through **A**, **B**, **C**. If the rect. **AB**,**AC** = the rect. **BC**,**CD**, prove that **AB** touches the circle through **A**, **C**, **D**.

THEOREM 7.

If a polygon is divided into triangles by lines joining a given point to its vertices, any similar polygon can be divided into corresponding similar triangles.



Let the polygon $ABCDE$ be divided into $\triangle^s$ OAB, OBC,...... by lines joining a given point O to its vertices.

Let $A'B'C'D'E'$ be a similar polygon, such that AB, $A'B'$; BC, $B'C'$; ...... are pairs of corresponding sides, and ABC, $A'B'C'$; BCD, $B'C'D'$; ...... are pairs of corresponding angles.

To prove that $A'B'C'D'E'$ *can be divided into triangles similar to the* $\triangle^s$ OAB, OBC, ...*by joining a certain point* O' *to its vertices.*

Construction. Make the $\angle B'A'O' =$ the $\angle BAO$, and the $\angle A'B'O' =$ the $\angle ABO$.

Join $O'C'$, $O'D'$, $O'E'$.

Proof. By construction, the $\triangle^s$ OAB, $O'A'B'$ are equiangular;

$$\therefore AB : A'B' = BO : B'O',$$

But the polygons $ABCDE$, $A'B'C'D'E'$ are similar ;

$$\therefore AB : A'B' = BC : B'C'$$

$$\therefore BO : B'O' = BC : B'C'.$$

Again the $\angle ABC =$ the $\angle A'B'C'$, [*Hyp.*

and the $\angle ABO =$ the $\angle A'B'O'$; [*Const.*

$$\therefore \text{the } \angle OBC = \text{the } \angle O'B'C'.$$

Now in the △ˢ OBC, O′B′C′,

the ∠OBC = the ∠O′B′C′,

and OB : O′B′ = BC : B′C′;

∴ the △ˢ OBC, O′B′C′ are similar. [V. 5.

Similarly it can be proved that the triangles OCD and O′C′D′; ODE and O′D′E′; OEA and O′E′A′ are similar.

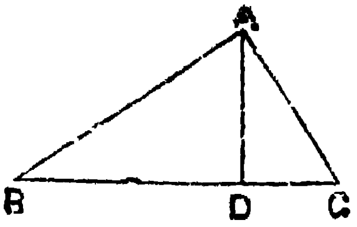
Q. E. D.

EXERCISES LXXVII.

Shew how to divide similar polygons into the same number of similar triangles by lines drawn from a pair of corresponding vertices.

*THEOREM 8. [Euc. VI. 8.]

In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.



Let ABC be a right-angled triangle having the ∠A a right angle ; and let AD be the perpendicular from A to BC.

To prove that *the* △ˢ DAC, DBA *are similar to* the △ABC, *and to one another.*

Proof. In the △ˢ DAC, ABC,

the ∠ADC = the ∠BAC (being rt. ∠ˢ),

and the ∠C is common ;

∴ the triangles are equiangular ;

∴ their corresponding sides are proportional ;

∴ the △ˢ DAC, ABC are similar.

In like manner, it can be proved that the △ˢ DBA, ABC are similar.

Also each of the △ˢ DAC, DBA is similar to the △ABC ;

∴ they are similar to one another. Q. E. D.

Cor. (i) AD is a mean proportional between CD and DB.

For △s **CDA, ADB** are similar, ∴ **CD : AD = AD : DB.**

Hence also $AD^2 = CD.DB$.

(ii) AB is a mean proportional between BD and BC.

For △s **DBA, ABC** being similar, **BD : AB = AB : BC.**

Hence also $AB^2 = BD.BC$.

(ii) AC is a mean proportional between CD and CB.

For △s **DCA, ACB** are similar, ∴ **CD : AC = AC : CB.**

Hence also $\mathbf{AC^2 = CD.CB}$.

EXERCISES LXXVIII.

[*Numerical and Theoretical.*]

N. B.—The answers are correct to one place of decimals.

1. In the fig. of Theorem 8, if **AC** = 6 in. and **CD** = 3 in., find **BC** and **AB**. Verify by measurement. [*Ans.* 12 in.; 10·4 in.

2. In the fig. of Theorem 8, if **AB** = 8 cm., and **BD** = 10 cm., find **BD** and **AD**. Verify by measurement. [*Ans.* 6·4 cm.; 4·8 cm.

3. In the fig. of Theorem 8, if **AD** = 1 8 cm. and **BC** = 4 cm., find **AB**, **AC**, **BD** and **CD**. Verify by measurement. [*Ans* 3·4 cm.; 2·1 cm.; 1·1 cm.

4. In the fig. of Theorem 8. if **CD** = 5 in. and **AB** : **AC** = 3 : 2, find **AD**. Verify by measurement [*Ans.* 7·5 in,

5. **If two circles touch externally, their common tangent is a mean proportional between their diameters.**

6. **Any tangent to a circle is drawn to cut a pair of parallel tangents; prove that the radius of the circle is a mean proportional between the segments into which the tangent is divided at the point of contact.**

7. **ABC** is a triangle right-angled at **A**, and **AD** is perpendicular to **BC**; if **AB** is a mean proportional between **AC** and **BC**, prove that **BC** is divided medially at **D.**

8. **In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the segments of the hypotenuse have the same ratio as the squares on the sides containing the right angle.**

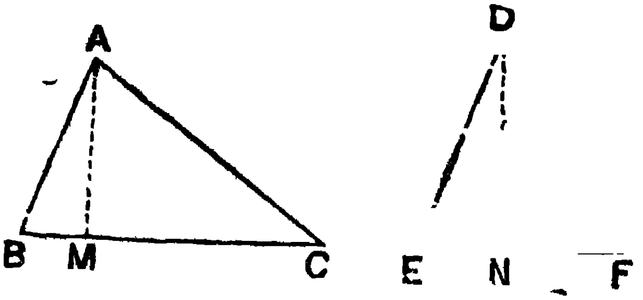
9. In the fig. of Theorem 8, if **AB** is twice **AD**, then **BC** is four times **CD.**

10. In a right-angled triangle, one side about the right angle is twice the other; if circles are described on these sides as diameters, prove that their common chord is two-fifths of the hypotenuse.

11. **Prove Pythagoras' Theorem by the help of this Theorem.**

*THEOREM 9.

If two triangles have one angle of the one equal to one angle of the other, the ratio of their areas is equal to the ratio of the rectangles contained by the sides about the equal angles.



Let ABC, DEF be two triangles, in which the ∠B = the ∠E.

To prove that △ ABC : △DEF = AB.BC : DE.EF.

Construction. Draw AM perp. to BC and DN to EF.

Proof. The △ ABC = ½ AM.BC,

and the △ DEF = ½ DN.EF ;

$$\therefore \frac{\triangle\ ABC}{\triangle\ DEF} = \frac{\frac{1}{2}\ AM.BC}{\frac{1}{2}\ DN.EF} = \frac{AM.BC}{DN.EF}.$$

Now because the ∠ B = the ∠ E,

and the ∠ AMB = the ∠DNE ;

∴ the △s AMB, DNE are equiangular ;

$$\therefore \frac{AM}{DN} = \frac{AB}{DE};$$

$$\therefore \frac{\triangle\ ABC}{\triangle\ DEF} = \frac{AB.BC}{DE.EF};$$

that is, △ ABC : △ DEF = AB.BC : DE.EF.

Q. E. D.

Cor. *If two parallelograms have one angle of the one equal to one angle of the other, the ratio of their areas is equal to the ratio of the rectangles contained by the sides about the equal angles.* [Euc. VI. 23.]

For either diagonal of a parallelogram divides it into two equal triangles.

[*ALTERNATIVE PROOF.*]

Since the $\triangle ABC = \frac{1}{2} AB.BC \sin B$,

and the $\triangle DEF = \frac{1}{2} DE.EF \sin E$,

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} AB.BC \sin B}{\frac{1}{2} DE.EF \sin E}.$$

But because the $\angle B =$ the $\angle E$; $\therefore \sin B = \sin E$.

$$\frac{\triangle ABC}{\triangle DEF} = \frac{AB.BC}{DE.EF}.$$

Q. E. D.

EXERCISES LXXIX.

1. **If two triangles have one angle of the one supplementary to one angle of the other, the ratio of their areas is equal to the ratio of the rectangles contained by the sides about these angles.**

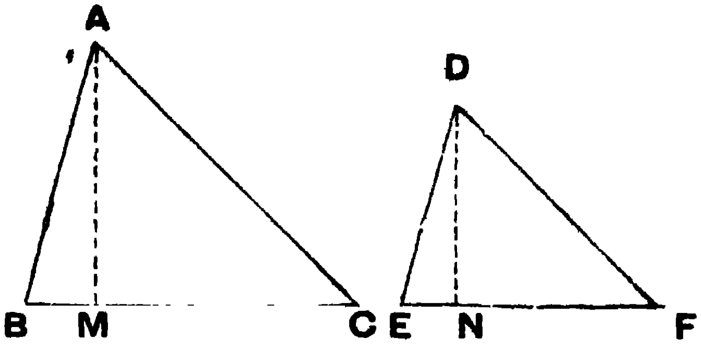
2. **ABC** is a triangle in which the $\angle$**A** is a right angle, and **AD** perpendicular to **BC**. Equilateral triangles **BCP**, **CAQ**, **ABR** are described on the sides. Prove that the $\triangle$**ABR** = the $\triangle$**BDP** and the $\triangle$**ACQ** = the $\triangle$**CDP**.

Hence shew that the $\triangle$**BCP** = the $\triangle$**ABR** + the $\angle$**AOQ**.

3. **From this Theorem, deduce a proof of Pythagoras' Theorem.**

THEOREM 10. [Euc. VI. 19.]

The ratio of the areas of two similar triangles is equal to the ratio of the squares on corresponding sides.



Let ABC, DEF be two similar triangles, in which the $\angle A = \angle D$, the $\angle B =$ the $\angle E$ and the $\angle C =$ the $\angle F$; so that BC, EF are corresponding sides.

To prove that $\triangle ABC : \triangle DEF = BC^2 : EF^2$.

Construction. Draw AM perp. to BC, and DN to EF.

Proof. Then the $\triangle ABC = \frac{1}{2}AM.BC$,

and the $\triangle DEF = \frac{1}{2}DN.EF$;

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}AM.BC}{\frac{1}{2}DN.EF} = \frac{AM.BC}{DN.EF}.$$

Now because the $\angle B =$ the $\angle E$, [*Hyp.*

and the $\angle AMB =$ the $\angle DNE$ (right angles);

$\therefore$ the $\triangle^s$ AMB, DNE are equiangular.

$$\therefore \frac{AM}{DN} = \frac{AB}{DE} = \frac{BC}{EF} \text{ from simlr. } \triangle^s \text{ ABC, DEF.}$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{AM.BC}{DN.EF} = \frac{BC.BC}{EF.EF} = \frac{BC^2}{EF^2};$$

that is, $\triangle ABC : \triangle DEF = BC^2 : EF^2$. Q. E. D.

Obs. Evidently, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB^2}{DE^2}$ or $\dfrac{AC^2}{DF^2}$ or $\dfrac{BC^2}{EF^2}$

EXERCISES LXXX.

[*Numerical and Theoretical.*]

1. Two similar triangles are on bases of 2 in. and 3 in. respectively; what is the ratio of their areas? [*Ans.* 4 : 9.

2. Two corresponding sides of two similar triangles are 4 cm. and 6 cm.; if the area of the first is 32 sq. cm., find the area of the second. [*Ans.* 72 sq. cm.

3. **ABC, DEF** are two similar triangles, of which the **△ABC** is 5 times the **△DEF**; if a side of the **△ABC** is 3 in., what is the length of the corresponding side of the **△DEF**? [*Ans.* 1·34 in.

4. The areas of two similar triangles are 486 sq. cm. and 384 sq. cm. If a side of the first is 9 cm., find the length of the corresponding side of the other. [*Ans.* 8 cm.

5. The straight line joining the middle points of the sides of a triangle cuts off a triangle equal to one-fourth of the given triangle.

6. Any triangle described on a diagonal of a square is double the similar triangle described on a side of the square.

7. **If two triangles are similar, their areas are proportional to the squares on (i) their corresponding medians, (ii) their in-radii and (iii) their circum-radii.**

8. **ABC** is a triangle; **PQ** is drawn parallel to **BC** through **P** a point of trisection of **AB** nearer to **A**. Compare the **△**s **APQ, ABC**.

9. **ABCD** is a cyclic quadrilateral of which the sides **DA, CB** are produced to meet in **O**. If **AB** is half of **CD**, prove that the quadrilateral **ABCD** is three times the **△OAB**.

10. Divide the **△ABC** by a line **PQ** parallel to **BC** and cutting **AB, AC** in **P, Q**, so that the **△APQ** may be $\frac{4}{9}$ of the **△ABC**.

[*Hint.* Trisect **AB**; and let **P** be a pt. of trisection of **AB** nearer to **B**.]

11. Bisect a triangle by a straight line parallel to one of the sides.

[*Const.* Let **ABC** be a **△**. Bisect **AB** at **D**; construct a sq. on **AD**; from **AB** cut off **AP** equal to a diagonal of this sq.; draw **PQ** parl. to **BC**.]

12. **If ABC is a triangle inscribed in a circle and the tangent at A meets BC produced in D; prove that CD : BD = AC² : AB².**

13. **ABC** is a triangle; on **BC** as diameter a circle is described cutting **AB**, **AC** in **P** and **Q**; prove that if the $\triangle$**APQ** is half of the $\triangle$**ABC**, then **PQ** is the side of a square that can be inscribed in the circle.

14. **ABC** is an acute-angled triangle and **D**, **E**, **F** are the feet of the altitudes. Prove that

(i) $\triangle ABC : \triangle AEF = AB^2 : AE^2$;

(ii) quadl. $BCEF : \triangle AEF = BE^2 : AE^2$.

15. **AB** is a diameter of a circle whose centre is **O**; from any point **C** on the circumference, **CD** is drawn perpendicular to **AB**; and **AE**, **CE** are tangents to the circle. Prove that

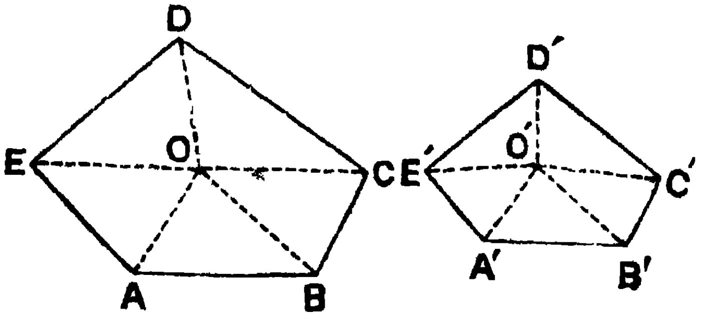
$$\triangle ECA : \triangle OCB = AD : DB.$$

16. Two circles intersect at **P** and **Q**, and at **P** a tangent is drawn to each meeting the circles at **A** and **B**; prove that

$$AQ : BQ = \triangle AQP : \triangle PQB.$$

THEOREM 11. [Euc. VI. 20.]

The ratio of the areas of two similar polygons is equal to the ratio of the squares on corresponding sides.



Let ABCDE, A′B′C′D′E′ be two similar polygons, in which angles A, B, C,.........are respectively equal to angles A′, B′, C′,...... ; so that AB, A′B′ are corresponding sides.

To prove that

polygon ABCDE : *polygon* $A'B'C'D'E' = AB^2 : A'B'^2$

Construction. Take a pt. O within ABCDE ; and divide it into △ˢ OAB, OBC,......Then A′B′C′D′E′ can be divided into corresponding similar △ˢ O′A′B′, O′B′C′......

[V. 7.

Proof. Because the polygons are divided into corresponding similar △ˢ,

$$\therefore \frac{\triangle OAB}{\triangle O'A'B'} = \frac{AB^2}{A'B'^2}, \quad \frac{\triangle OBC}{\triangle O'B'C'} = \frac{BC^2}{B'C'^2} \cdots\cdots\cdots,$$

$$\frac{\triangle OEA}{\triangle O'E'A'} = \frac{EA^2}{E'A'^2}.$$

And because the polygons are similar ;

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \cdots\cdots = \frac{EA}{E'A'} ;$$

$$\therefore \frac{\triangle OAB}{\triangle O'A'B'} = \frac{\triangle OBC}{\triangle O'B'C'} = \ldots\ldots = \frac{\triangle OEA}{\triangle O'E'A'};$$

$$\therefore \frac{\triangle OAB + \triangle OBC + \ldots\ldots + \triangle OEA}{\triangle O'A'B' + \triangle O'B'C' + \ldots\ldots + \triangle O'E'A'}$$

$$= \frac{\triangle OAB}{\triangle O'A'B'} = \frac{AB^2}{A'B'^2}. \qquad [\textit{Lem. } 13.$$

$\therefore$ polygon $ABCDE$: polygon $A'B'C'D'E' = AB^2 : A'B'^2$.

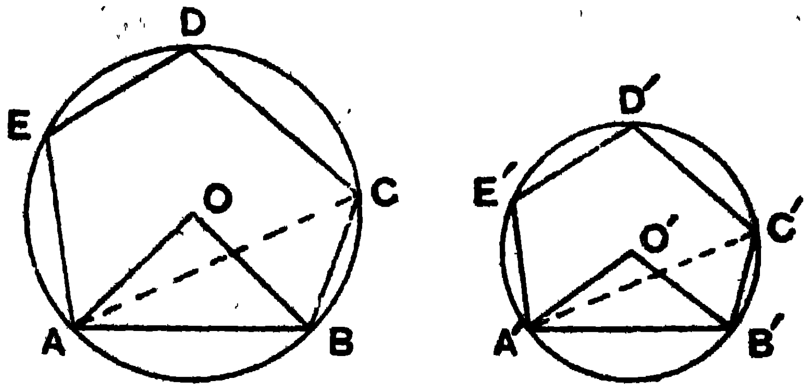
Q. E. D.

EXERCISE LXXXI.

If three straight lines are proportionals, the first is to the third as any polygon described on the first is to a similar and similarly described polygon on the second. [Use *Lem.* 7.]

*THEOREM 12.

Similar polygons inscribed in circles are to one another as the squares on the radii of the circles.



Let ABCDE and A′B′C′D′E′ be similar polygons inscribed in circles of centres O, O′ and radii OA, O′A′.

Let AB, A′B′ ; BC, B′C′ ;...... be corresponding sides.

To prove that

polygon ABCDE : *polygon* A′B′C′D′E′ = OA^2 : $O'A'^2$.

Construction. Join OB, O′B′, AC, A′C′.

Proof. Because the polygons are similar,

∴ the ∠ABC = the ∠A′B′C′,

and AB : BC = A′B′ : B′C′ ; [*Hyp.*

∴ the △s ABC, A′B′C′ are similar, [V. 5.

so that the ∠ACB = the ∠A′C′B′ ;

∴ the ∠AOB = the ∠A′O′B′ ; [III. 5.

Again the ∠AOB = the ∠A′O′B′ ;

∴ their supplements are equal ;

∴ 2 ∠OAB = 2 ∠O′A′B′ ; *i.e.*, the ∠OAB = the ∠O′A′B′.

∴ the △s OAB, O′A′B′ are equiangular ;

∴ AB : A′B′ = OA : O′A.

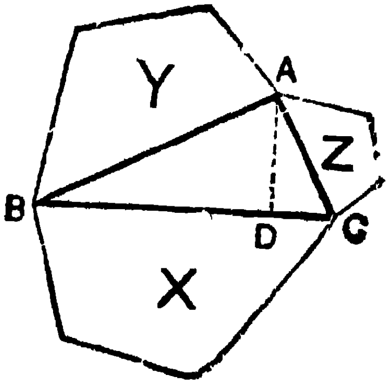
But polygon ABC... : polygon A′B′C′... = AB^2 : $A'B'^2$. [V. 11.

∴ polygon ABCDE : polygon A′B′C′D′E′ = OA^2 : $O'A'^2$.

Q. E. D.

*THEOREM 18. [Euc. VI. 31.]

In a right-angled triangle, any polygon described on the hypotenuse is equal to the sum of the similar polygons similarly described on the sides containing the right angle.



Let ABC be a right-angled triangle in which BC is the hypotenuse.

Let X, Y and Z be similar polygons similarly described on BC, AB and AC respectively.

To prove that $X = Y + Z$.

Construction. Draw AD perp. to BC.

Proof. Then the $\triangle^s$ DBA, ABC are similar ; [V. 8.

$$\therefore BD : AB = AB : BC ;$$

$$\therefore \frac{BD}{BC} = \frac{AB^2}{BC^2}. \qquad [\textit{Lem. } 7.$$

But $\because$ Y and X are similar polygons ;

$$\therefore \frac{Y}{X} = \frac{AB^2}{BC^2} \text{ [V. 11]} = \frac{BD}{BC}.$$

Similarly from the similar $\triangle^s$ DAC, ABC, it may be shewn that

$$\frac{Z}{X} = \frac{AC^2}{BC^2} = \frac{DC}{BC}.$$

$$\therefore \frac{Y}{X}+\frac{Z}{X}=\frac{BD}{BC}+\frac{DC}{BC};$$

that is, $$\frac{Y+Z}{X}=\frac{BD+DC}{BC}=1.$$

$$\therefore X=Y+Z.$$

Q. E. D.

[ALTERNATIVE PROOF.]

Proof. Since X and Y are similar figures ;

$$\therefore \frac{Y}{X}=\frac{AB^2}{BC^2}.$$ [V. 11.

Similarly $$\frac{Z}{X}=\frac{AC^2}{BC^2}.$$

$$\therefore \frac{Y}{X}+\frac{Z}{X}=\frac{AB^2}{BC^2}+\frac{AC^2}{BC^2};$$

that is, $$\frac{Y+Z}{X}=\frac{AB^2+AC^2}{BC^2}=1;$$

because $AB^2+AC^2=BC^2$, [II. 5.

$$\therefore X=Y+Z.$$

Q. E. D.

Note. The Theorem of Pythagoras [II. 5] is a particular case of this general Theorem.

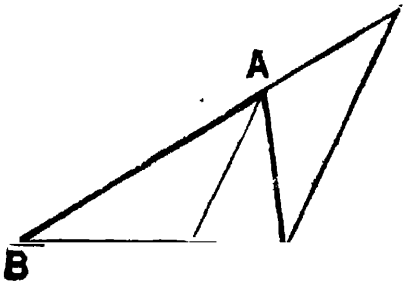
EXERCISE LXXXII.

Construct (i) a square equal to the sum of two given squares ; (ii) a rectangle equal to the sum of two similar rectangles ; and (iii) an equilateral triangle equal to the sum of two given equilateral triangles.

THEOREM 14. [Euc. VI. 3.]

(i) The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(ii) Conversely, if a straight line drawn from an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, it is the internal bisector of that angle.



Let ABC be a triangle.

(i) Let AD bisect the $\angle BAC$ internally and cut BC at D.

To prove that $BD : DC = BA : AC$.

Construction. Through C, draw CE ∥ to AD, meeting BA produced at E.

Proof. Because AD is ∥ to EC,

$$\therefore \left\{\begin{array}{l}\text{the } \angle BAD = \text{the corresponding } \angle AEC,\\ \text{and the } \angle DAC = \text{the alternate } \angle ACE.\end{array}\right.$$

But the $\angle BAD =$ the $\angle DAC$; [*Hyp.*

$\therefore$ the $\angle AEC = \angle ACE$;

$\therefore AE = AC$.

Now, because AD is ∥ to EC, a side of the $\triangle BCE$;

$\therefore BD : DC = BA : AE$; [V. 2.

that is, $BD : DC = BA : AC$.

(ii) *Conversely*, let AD divide BC internally at D, so that BD : DC = BA : AC.

To prove that AD *is the internal bisector of the* ∠A.

Construction. Through C, draw CE ∥ to AD, meeting BA produced at E.

Proof. Because AD is ∥ to EC, a side of the △BCE,

∴ BD : DC = BA : AE.

But BD : DC = BA : AC; [*Hyp.*

∴ BA : AE = BA : AC;

∴ AE = AC;

∴ the ∠AEC = the ∠ACE.

Again, because AD is ∥ to EC;

$$\therefore \begin{cases} \text{the } \angle BAD = \text{the corresponding } \angle AEC, \\ \text{and the } \angle DAC = \text{the alternate } \angle ACE; \end{cases}$$

∴ the ∠BAD = the ∠DAC;

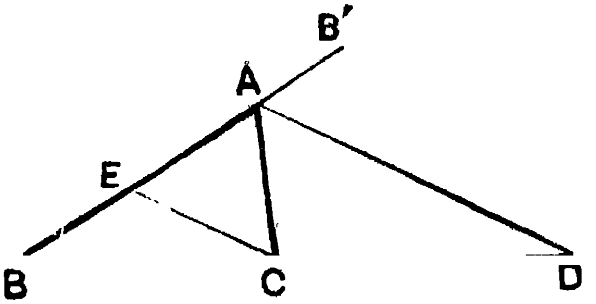
that is, AD bisects the ∠A internally.

Q. E. D.

THEOREM 15.

(i) The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.

(ii) Conversely, if a straight line drawn from an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle, it is the external bisector of that angle.



Let ABC be a triangle.

(i) Let AD bisect the ∠CAB′ and cut BC produced at D; then AD is the external bisector of the ∠BAC.

[I. 1, *Def.* 3.]

To prove that BD : DC = BA : AC.

Construction. Through C, draw CE ∥ to AD, meeting BA at E.

Proof. Because AD is ∥ to EC,

$\therefore\begin{cases}\text{the } \angle B'AD = \text{the corresponding } \angle AEC, \\ \text{and the } \angle DAC = \text{the alternate } \angle ACE.\end{cases}$

But the ∠B′AD = the ∠DAC. [*Hyp.*

∴ the ∠AEC = the ∠ACE;

∴ AE = AC.

Now because AD is ∥ to EC, a side of the △BCE;

∴ BD : DC = BA : AE; [V. 2.

that is, BD : DC = BA : AC.

(ii) *Conversely*, let AD, drawn from the $\angle A$, divide BC externally at D, so that $BD : DC = BA : AC$.

To prove that AD *is the external bisector of the* $\angle A$.

Construction. Through C, draw CE ∥ to AD, meeting BA at E.

Proof. Because AD is ∥ to EC, a side of the $\triangle BCE$;

$\therefore BD : DC = BA : AE$. [V. 2.

But $BD : DC = BA : AC$; [*Hyp.*

$\therefore BA : AE = BA : AC$;

$\therefore AE = AC$.

$\therefore$ the $\angle AEC =$ the $\angle ACE$.

Again because AD is ∥ to EC;

$$\therefore \left\{ \begin{array}{l} \text{the } \angle B'AD = \text{the corresponding } \angle AEC, \\ \text{and the } \angle DAC = \text{the alternate } \angle ACE. \end{array} \right.$$

$\therefore$ the $\angle B'AD =$ the $\angle DAC$;

that is, AD bisects the $\angle A$ externally.

Q. E. D.

EXERCISES LXXXIII.

1. Prove by Theorem 14 that if a straight line bisects the vertical angle as well as the base of a triangle, the triangle is isosceles.

2 Prove by Theorem 14 that the internal bisectors of the angles of a triangle are concurrent.

3. Prove by Theorems 14 and 15 that the bisector of the vertical angle of a triangle and the bisectors of the exterior angles at the base are concurrent.

4. **D** is the middle point of **BC**, the base of a triangle **ABC**; **DE**, **DF** bisect that angles **ADB**, **ADC** meeting **AB**, **AC** respectively in **E**, **F**. Shew that **EF** is parallel to **BC**.

5. The angle **A** of a triangle **ABC** is bisected by **AD**, meeting **BC** in **D**; the angle **B** is bisected by **BI**, meeting **AD** in **I**; prove that

$$AI : ID = AB + AC : BC.$$

6. The angle **A** of a triangle **ABC** is bisected by **AD**, meeting **BC** in **D**; **O** is the middle point of **BC**; prove that

$$OB : OD = AB + AC : AB - AC.$$

7. From any point **P** on a circle whose diameter is **AB**, **PC** and **PD** are drawn making equal angles with **AP** on opposite sides and meeting **AB** in **C** and **D**. Prove that $AD : DB = AC : CB$.

8. In a quadrilateral, if the bisectors of one pair of opposite angles meet on one diagonal, then the bisectors of the other pair of opposite angles will meet on the other diagonal.

9. **AB** is a fixed straight line and **C** is a fixed point in it; through **C**, any straight line **CD** is drawn; find a point **P** in **CD** such that

$$AP : PB = AC : CB.$$

[*Hint.* From **A** draw $AM \perp CD$, produce **AM** making $MN = AM$, join **NB** and produce **NB** to meet **CD** in **P**.]

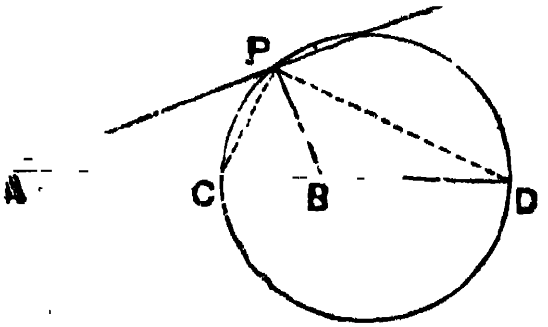
10. Construct a triangle, having given the base, the ratio of the sides and the vertical angle.

11. The internal and external bisectors of the vertical angle **APB** of a triangle cut the base **AB** in **C** and **D** respectively; if **O** is the middle point of **AB**, prove that $OB^2 = OC.OD$.

12. **The locus of a point which moves so that the ratio of its distances from two fixed points is constant is a circle.** [This locus is known as the *Circle of Apollonius.*]

Let **A** and **B** be two fixed pts., and let a pt. **P** move, so that $AP : PB$ is constant.

Let **P** be any position of the moving pt. Draw the two bisectors of the $\angle APB$ meeting **AB** in **C** and **D** respectively.



Then $AC : BC = AP : BP$

$= \text{const.};$

$\therefore$ **C** is a fixed pt. $\because$ **A** and **B** are fixed pts.

Similarly $\because AD : BD = AP : BP = \text{const.};$

$\therefore$ **D** is a fixed pt.

$\therefore$ **CD** is a st. line of fixed length.

Also the $\angle CPD$, being between the internal and external bisectors of the $\angle APB$, is a right angle.

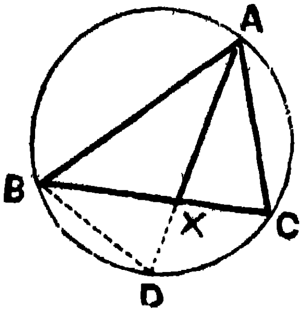
Hence the fixed line **CD** always subtends a right angle at **P**.

$\therefore$ the locus required is a circle described on **CD** as diameter.

Note. Evidently the ratio $AP : BP$ is supposed to be one of *inequality*; for when the ratio $= 1$, the required *locus is the perpendicular bisector of* **AB**. [II. 7.]

*THEOREM 16.

If the internal bisector of the vertical angle of a triangle meets the base, the rectangle contianed by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the bisector.



Let ABC be a triangle; and let AX be the internal bisector of the ∠ A, meeting BC at X.

To prove that $AB.AC = BX.XC + AX^2$.

Construction. Circumscribe a circle about the △ABC; and produce AX to meet the circle at D.

Join BD.

Proof. In the △^{s} BAD, XAC,

the ∠BAD = the ∠XAC, [*Hyp.*

and the ∠D = the ∠C in the same segment;

∴ the △BAD is equiangular to the △ XAC;

∴ AB : AX = AD : AC;

$$\therefore AB.AC = AX.AD$$
$$= AX(XD + AX)$$
$$= AX.XD + AX^2$$
$$= BX.XC + AX^2$$ [IV. 11.

Q. E. D.

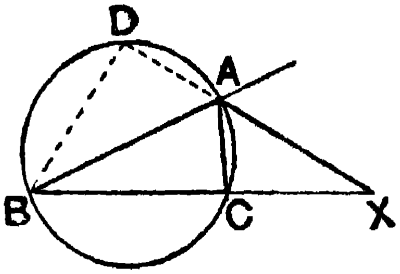
Cor. *If the internal bisector of the vertical angle* A *of a triangle meets the base* BC *in* X *and the circum-circle in* D, *then*

$$AB.AC = AX.AD.$$

Note. The following case, in which **AX** is the external bisector of the $\angle$**A** should be noticed.

If the external bisector of the vertical angle of a triangle meets the base produced, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base diminished by the square on the bisector.

Let **ABC** be a $\triangle$; and let **AX** be the external bisector of the $\angle$ **A**, meeting **BC** produced at **X**.



To prove that $AB.AC = BX.XC - AX^2$.

Construction. Produce **XA** to meet the circumcircle at **D**. Join **BD**.

Proof. Now the $\angle$ **BAD** = the vert. opp. angle = the $\angle$**XAC**,

and the $\angle$ **ADB** = the $\angle$**ACX**; [III. 9, *Cor.*

$\therefore$ the $\triangle$s **BAD, XAC** are equiangular;

$\therefore AB : AX = AD : AC.$

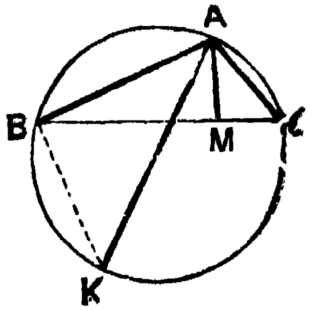
Hence $AB.AC = AX.AD = AX\,(DX - AX)$

$= DX.XA - AX^2 = BX.XC - AX^2.$ [IV. 11.

Q. E. D

*THEOREM 17.

If from the vertical angle of a triangle a perpendicular is drawn to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle. [Brahmagupta's Theorem.]



Let ABC be a triangle; and let AM be the perpendicular from A to the base BC, and AK a diameter of the circum-circle.

To prove that AB.AC = AM.AK.

Construction. Join BK.

Proof. In the Δ^s BAK, MAC,

the ∠K = the ∠C in the same segment,

and the ∠ABK in a semi-circle = a rt. ∠ = the rt. ∠AMC;

∴ the Δ BAK is equiangular to the Δ MAC;

∴ AB : AM = AK : AC;

that is, AB.AC = AM.AK.

Q. E. D.

Cor. In any triangle,

$$\text{diameter of circum-circle} = \frac{abc}{2\Delta}.$$

where Δ represents the area of the triangle ABC, and *a*, *b*, *c*, the sides opposite the angles A, B and C respectively.

$$\text{For } AK = \frac{AB.AC}{AM} = \frac{bc}{AM} = \frac{abc}{a.AM} = \frac{abc}{2\Delta}.$$

EXERCISES LXXXIV.

1. Prove that the rectangle contained by the sides of a triangle is generally greater than twice the area of the triangle. When are they equal?

2. In the figs. of V. 16 and the Note, prove that BD is a mean proportional to AD and DX.

Hence shew that BD is a tangent to the circum-circle of the $\triangle ABX$.

3. X is any point in the base BC of a triangle ABC. Prove that the radii of the circum-circles of triangles ABX, ACX are in the ratio of AB to AC.

4. Triangles ABC, ABD having the same base AB are inscribed in the same circle. Prove that the rectangles contained by their sides are proportional to their altitudes.

5. The diagonal AC of a cyclic quadrilateral $ABCD$ is bisected by BD. Shew that the rect. $AB,AD =$ the rect. CB,CD.

6. If P is any point on the arc AB of a circle, prove that the rectangle contained by the chords PA, PB is greatest when $PA = PB$.

7. The diagonals of a cyclic quadrilateral $ABCD$ intersect at O. Prove that

$$\text{rect. } AB,AD : \text{rect. } CB,CD = OA : OC.$$

8. Triangles APB, AQB are inscribed in the same circle, so that P, Q are on the same side of AB; O is the middle point of arc AB; PO, QO cut the base at C and D If $AC = BD$, shew that

$$\text{the rect. } PA,PB + QD^2 = \text{the rect. } QA,QB + PC^2.$$

9. From A, a tangent AT and a secant APQ are drawn to a circle. Prove that the radii of the circum-circles of the $\triangle$s APT, AQT are in the ratio $AP : AT$.

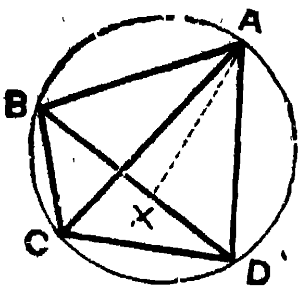
10. O is the centre of a fixed circle, and a tangent to this circle cuts another fixed circle passing through O in X and Y. Prove that the rect. OX,OY is constant.

11. In a cyclic quadrilateral $ABCD$, prove that

$$AB.AD + CB.CD : BA.BC + DA.DC = AC : BD.$$

*THEOREM 18.

The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the two rectangles contained by its opposite sides. [Ptolemy's Theorem.]



Let ABCD be a cyclic quadrilateral; and let AC, BD be its diagonals.

To prove that AC.BD = AB.CD + BC.AD.

Construction. Make the ∠DAX = the ∠BAC.

Proof. Now the ∠BAC = the ∠XAD. [*Const.*

to each, add the ∠CAX,

∴ the ∠BAX = the ∠CAD ;

also the ∠ABX = the ∠ACD in the same segment ;

∴ the △BAX is equiangular to the △CAD ;

∴ AB : AC = BX : CD ;

∴ AB.CD = AC.BX..................(i).

Again, the ∠BAC = the ∠XAD, [*Const.*

and the ∠ACB = the ∠ADX in the same segment,

∴ the △ABC is equiangular to the △AXD ;

∴ BC : XD = AC : AD ;

∴ BC.AD = AC.XD..................(ii).

Add (i) and (ii) ; thus

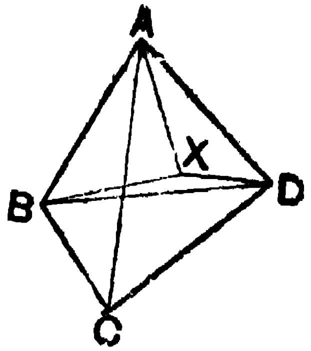
$$\begin{aligned} AB.CD+BC.AD &= AC.BX+AC.XD \\ &= AC(BX+XD) \\ &= AC.BD. \end{aligned}$$

Q. E. D.

Note. If the quadrilateral is not cyclic, we have the following

Theorem. The rectangle contained by the diagonals of any quadrilateral is less than the sum of the two rectangles contained by its opposite sides.

Let $ABCD$ be a quadrilateral, in which AC, BD are diagonals.



Construction. Make the $\angle DAX$ = the $\angle BAC$ and the $\angle ADX$ = the $\angle ACB$. Let AX, DX meet at X.

Join BX.

Proof. By *Const.*, the Δs ABC, AXD are equiangular ;

$\therefore$ $BC : AC = XD : AD$

or $BC.AD = AC.XD$..........................(i).

Again, the $\angle BAX$ = the $\angle CAD$;

and from similar Δs ABC, AXD,

$AB : AX = AC : AD$;

$\therefore$ the Δs ABX, ACD are similar. [V. 5.

$\therefore$ $AB : BX = AC : CD$;

or $AB.CD = AC.BX$..................................(ii)

Add (i) and (ii) ; thus

$$\begin{aligned} AB.CD+BC.AD &= AC.BX+AC.XD \\ &= AC(BX+XD). \end{aligned}$$

But $BX+XD > BD$;

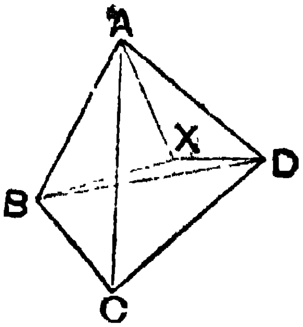
$\therefore$ $AB.CD+BC.AD > AC.BD$.

Q. E. D.

*THEOREM 18A.

[CONVERSE OF THEOREM 18.]

If the rectangle contained by the diagonals of a quadrilateral is equal to the sum of the rectangles contained by its opposite sides, the quadrilateral can be inscribed in a circle.



Let ABCD be a quadrilateral, in which AC, BD are diagonals.

Let AC.BD=AB.CD+BC.AD.

To prove that *the quadrilateral* ABCD *can be inscribed in a circle.*

Construction. Make the ∠DAX = the ∠BAC,
and the ∠ADX = the ∠ACB.

Let AX, DX meet at X. Join BX.

Proof. Then the △ˢ ABC, AXD are equiangular ;

∴ BC : AC=XD : AD ; or BC.AD = AC.XD......(i).

Again the ∠BAX = the ∠CAD ;
and from similar △ˢ ABC, AXD,

AB : AX=AC : AD ;

∴ the △ˢ ABX, CAD are similar. [V. 5.

Hence AB : BX=AC : CD ; or AB.CD = AC.BX...(ii)

From (i) and (ii), AB.CD + BC.AD = AC.BX + AC.XD
= AC(BX + XD).

By *Hyp.*, $AB.CD + BC.AD = AC.BD$;

$\therefore BD = BX + XD.$

Thus X is a pt. in BD, and the ∠ADX coincides with the ∠ADB.

But the ∠ADX = the ∠ACD; [*Const.*

∴ the ∠ADB = the ∠ACB.

Hence AB subtends equal angles at C and D;

∴ A, B, C, D are concyclic. [III. 7.

Q. E. D.

EXERCISES LXXXV.

1. The distances of any point on the circum-circle of an equilateral triangle from its vertices are such that one of them is equal to the sum of the other two.

2. **AB** is a fixed chord of a circle, and **C** the middle point of the arc **AB**. If **P** is any point on the circle, prove that the ratio

$$PA + PB : PC \text{ is constant.}$$

3. If the diagonals of a cyclic quadrilateral are at right angles, the sum of the rectangles contained by its opposite sides is equal to twice the area of the quadrilateral.

4. A regular pentagon **ABCDE** is inscribed in a circle, and **P** is any point on the arc **AE**. Prove that

$$PA + PC + PE = PB + PD.$$

5. From **A**, tangents **AB, AC** are drawn to a circle, and a secant **ADE** cuts the circle at **D** and **E**. Prove that in the quadrilateral **BDCE**,

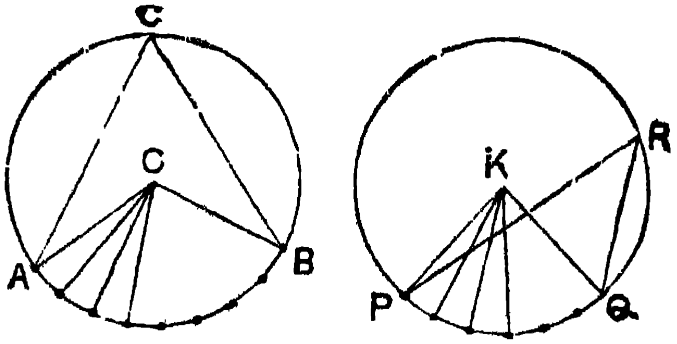
$$BC.DE = 2BD.CE.$$

6. **ABCD** is a parallelogram and any circle through **A** cuts **AB, AD** and **AC** in **P, Q** and **R** respectively. Prove that

$$AP.AB + AQ.AD = AR.AC.$$

*THEOREM 19. [Euc. VI. 33.]

In equal circles, angles at the centres or at the circumferences have the same ratio as the arcs on which they stand.



Let ABC, PQR be two equal circles whose centres are O and K; and let the angles AOB, PKQ at the centres, and the angles ACB, PRQ at the circumferences, stand on the arcs AB, PQ.

To prove that

(i) *the* $\angle$ AOB : *the* $\angle$ PKQ = *the arc* AB : *the arc* PQ;

(ii) *the* $\angle$ ACB : *the* $\angle$ PRQ = *the arc* AB : *the arc* PQ.

(i)

Proof. Let the arcs AB and PQ have a common measure m.

Then each of the arcs AB, PQ can be divided into equal parts, each of which is m.

Let the arc AB contain p and the arc PQ contain q of these parts; so that the arc AB $= pm$, and the arc PQ $= qm$.

Because in equal circles, equal arcs subtend equal angles at the centres;

∴ the $\angle^s$ AOB, PKQ are divided into p and q angles, *all* equal to one another. Let each of these angles $= \alpha$.

Thus the $\angle AOB = pa$, and the $\angle PKQ = qa$.

∴ the $\angle AOB$: the $\angle PKQ = pa : qa = p : q$.

But the arc AB : the arc PQ $= pm : qm = p : q$.

∴ the $\angle AOB$: the $\angle PKQ$ = the arc AB : the arc PQ.

(ii)

Again the $\angle ACB = \frac{1}{2} \angle AOB$, and the $\angle PRQ = \frac{1}{2} \angle PKQ$;

∴ the $\angle ACB$: the $\angle PRQ$ = the arc AB : the arc PQ.

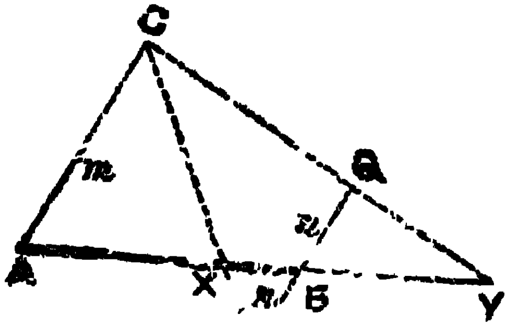
Q. E. D.

Cor. *In equal circles, sectors have the same ratio as the arcs on which they stand.*

This can be proved as above by the help of III. 12. *Cor.*

CONSTRUCTIONS

Problem 1. To divide a given straight line internally and externally in a given ratio ($m : n$).



Let AB be the given straight line.

To divide AB *internally and externally in the ratio* $m : n$.

Construction. Draw AC making any angle with AB.

Let AC $= m$ units of length.

Through B, draw PQ || to AC ; and let BP = BQ $= n$ units of length. Join CP, CQ. Let CP cut AB in X, and CQ cut AB produced in Y.

Then AB is divided internally at X and externally at Y in the ratio $m : n$.

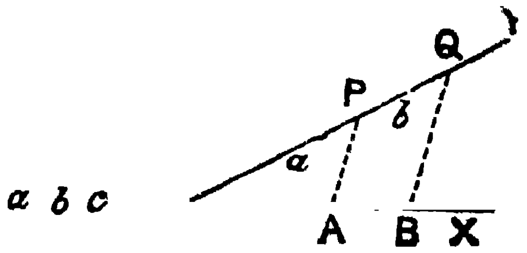
Proof. Since AC is || to PQ ;

$$\therefore \begin{cases} \text{the } \triangle \text{ ACX is similar to the } \triangle \text{BPX;} \\ \text{the } \triangle \text{ AYC is similar to the } \triangle \text{BYQ.} \end{cases}$$

$$\therefore \begin{cases} \text{AX : XB = AC : BP} = m : n \text{; and} \\ \text{AY : YB = AC : BQ} = m : n. \end{cases}$$

Obs. We have supposed $m > n$. But if $m < n$, then Y would be in BA produced through A. The student will do well to draw the figure to satisfy himself.

Problem 2. **To find the fourth proportional to three given straight lines.**



Let a, b, c be the three given straight lines.

To find the fourth proportional to a, b and c.

Construction. Draw any two st. lines making an ∠XOY.

From OY, cut off OP $= a$ and PQ $= b$. From OX, cut off OA $= c$. Join PA. Through Q, draw QB ∥ to PA cutting OX in B.

Then AB is the fourth proportional required.

Proof. In the △OBQ, ∵ AP is ∥ to BQ,

$$\therefore \text{OP} : \text{PQ} = \text{OA} : \text{AB};$$

that is, $a : b = c : \text{AB}$.

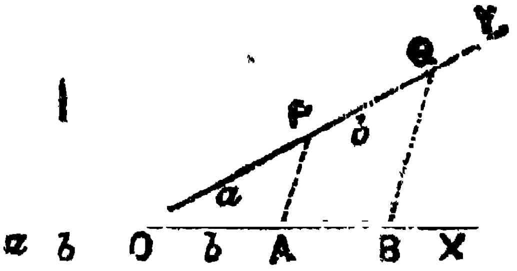
Thus AB is the fourth proportional to a, b and c.

Problem 3. **To find the third proportional to two given straight lines.**

Let a, b be the two given straight lines.

To find the third proportional to a and b.

FIRST METHOD



Construction. Draw any two st. lines making an ∠XOY.

From OY, cut off OP $= a$ and PQ $= b$. From OX, cut off OA $= b$. Join PA. Through Q, draw QB ∥ to PA, cutting OX in B.

Then AB is the third proportional required.

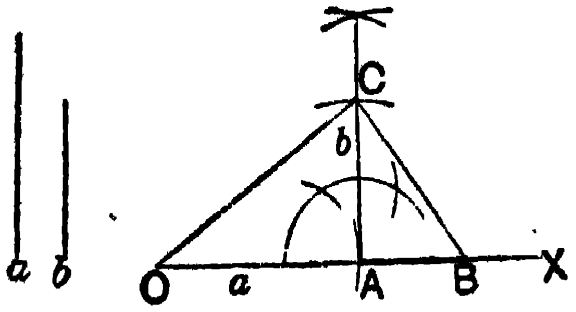
Proof. In the △ OBQ, ∵ AP is ∥ to BQ,

∴ OP : PQ = OA : AB ;

that is, $a : b = b :$ AB.

Thus AB is the third proportional to a and b.

SECOND METHOD



Construction. Take any st. line OX. Make OA $= a$. Draw AC perp. to OX. Make AC $= b$. Join OC ; and draw CB at rt. angles to OC, cutting OX in B.

Then AB is the third proportional required.

Proof. Since CA is perp. from the rt. ∠C to OB ;

∴ the △ˢ OAC, CAB are similar ;

hence OA : AC = AC : AB ;

that is, $a : b = b :$ AB.

Thus AB is the third proportional to a and b.

Note. If c be a third proportional to a and b,

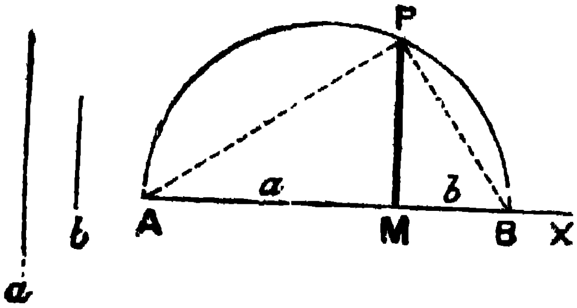
then $$ac = b^2.$$

Problem 4. To find the mean proportional between two given straight lines.

FIRST METHOD

Let a, b be the two given straight lines.

To find the mean proportional between a *and* b.



Construction. Take any st. line AX. Make AM $= a$, MB $= b$. On AB as diameter, draw a semi-circle APB. Draw MP perp. to AB, to cut the circumference in P.

Then PM is the mean proportional required.

Proof. Join AP, BP.

Then the ∠ APB in a semi-circle is a right angle.

Thus in the right-angled △ APB, PM is drawn perp. from the right angle to the hypotenuse AB ;

∴ the △s AMP, PMB are similar ; [V. 8.

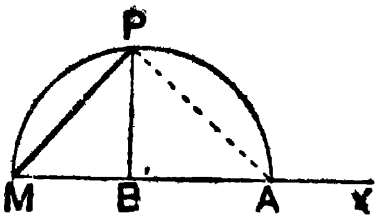
∴ AM : PM = PM : MB ; that is, a : PM = PM : b.

Thus PM is the mean proportional between a and b.

SECOND METHOD

Let a, b be the two given straight lines.

To find the mean proportional between a *and* b.



Construction. Take any st. line **MX**. Make **MA** $= a$, **MB** $= b$. On **MA** as diameter, draw a semi-circle **APM**. Draw **BP** perp. to **MA** to cut the circumference in **P**.

Join **PM**.

Then **PM** is the mean proportional required.

Proof. Join **AP**.

Then the $\angle$ **APM** in a semi-circle is a right angle.

Thus in the right-angled $\triangle$ **APM**, **PB** is drawn from the right angle perp. to the hypot. **AM** ;

∴ the $\triangle^s$ **AMP**, **PMP** are similar ; [V. 8.

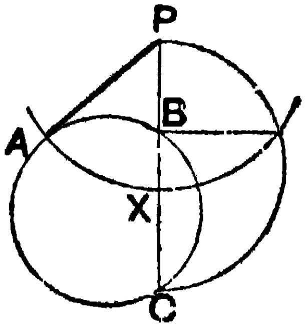
∴ **AM** : **PM** = **PM** : **MB** ; that is, a : **PM** = **PM** : b.

Hence **PM** is the mean proportional between a and b.

Note 1. From the figure, it is evident that **PM** $< a$ but $> b$. Thus **the mean proportional between two quantities lies in magnitude between them.**

Note 2. From the above we have $PM^2 = ab$, *i.e.*, $PM = \sqrt{(ab)}$. Thus **the mean proportional between two quantities is the square root of their product.**

Note 3. **To draw a tangent to a given circle ABC from a given external point P.**



Construction. From **P**, draw any secant **PBC** to cut the circle at **B**, **C**.

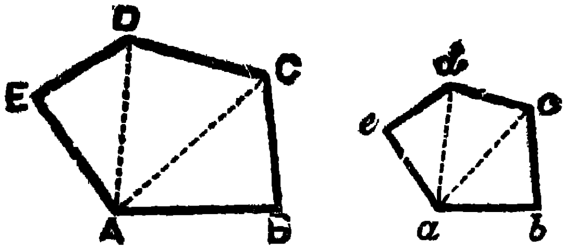
Take **PX** the mean proportional to **PB** and **PC**. With centre **P** and radius **PX**, draw a circle to cut the given ⊙ at **A**. Join **PA**.

Then **PA** is the required tangent.

Proof. Now $PB.PC = PX^2 = PA^2$;

∴ **PA** is a tangent to the circle **ABC**. [IV. 13.

Problem 5. On a given straight line, to construct a polygon similar to a given polygon.



Let *ab* be the given st. line and ABCDE the given polygon.

To construct a polygon on ab similar to ABCDE.

Construction. Join AC, AD; and place *ab* || to AB. Draw *bc* parallel to BC, *ac* to AC, *cd* to CD, *ad* to AD, *de* to DE and *ae* to AE.

Then *abcde* is the polygon required.

Proof. Because *ab* is || to AB, *bc* to BC, *cd* to CD, etc.

∴ ∠*abc* = ∠ABC, ∠*bcd* = ∠BCD, ∠*cde* = ∠CDE, etc.

Thus the two polygons are equiangular.

Again because *ab* is || to AB, *bc* to BC and *ac* to AC;

∴ the Δ^s *abc*, ABC are equiangular;

$$\text{so that } \frac{ac}{AC} = \frac{ab}{AB} = \frac{bc}{BC},$$

For a similar reason, the Δ^s *acd*, ACD are equiangular;

$$\text{so that } \frac{ac}{AC} = \frac{cd}{CD},$$

$$\therefore \frac{ab}{AB} = \frac{bc}{BC} = \frac{cd}{CD}.$$

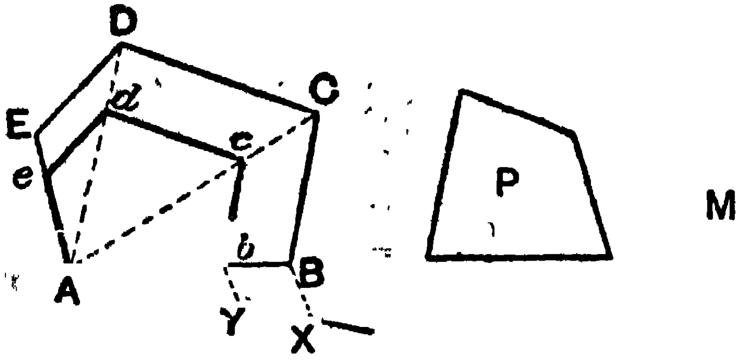
Similarly from the Δ^s *acd*, ACD and the Δ^s *ade*, ADE, it may be shewn that $\frac{cd}{CD} = \frac{de}{DE} = \frac{ea}{EA}$.

$$\therefore \frac{ab}{AB} = \frac{bc}{BC} = \frac{cd}{CD} = \frac{de}{DE} = \frac{ea}{EA}.$$

Thus the sides of the polygons are proportionals.

Hence the polygons *abcde*, ABCDE are similar.

Problem 6. To construct a polygon similar to one given polygon and equal in area to another given polygon.



Let ABCDE and P be the two given polygons.

To construct a polygon similar to ABCDE *and equal to* P.

Construction. Construct squares equal in area to ABCDE and P respectively. [IV. *Prob.* 5, *Note.*

Let L and M be the sides of these squares.

Draw any st. line AYX through A ;

and make AX = L and AY = M.

Join XB, AC, AD.

Draw Y*b*, *bc*, *cd*, *de* parallel to XB, BC, CD, DE respectively, to cut AB, AC, AD, AE in *b*, *c*, *d* and *e*.

Then A*bcde* is the polygon required.

Proof. Because the △ˢ ABC, A*bc* ; ACD, A*cd* ; and ADE, A*de* are similar ;

∴ polygon A*bcde* is similar to polygon ABCDE.

And $\dfrac{\text{polygon P}}{\text{polygon ABCDE}} = \dfrac{AY^2}{AX^2} = \dfrac{Ab^2}{AB^2} = \dfrac{\text{polygon } Abcde}{\text{polygon ABCDE}}$

∴ polygon P = polygon A*bcde*.

Def. A straight line AB is said to be divided *in extreme and mean ratio* at M, when AB : AM = AM : BM.

Problem 7. To divide a given straight line in extreme and mean ratio.

Let AB be the given straight line.

To divide it in extreme and mean ratio.

Divide AB at M so that $AB.BM = AM^2$. [IV. *Prob.* 3.

Hence $AB : AM = AM : BM$.

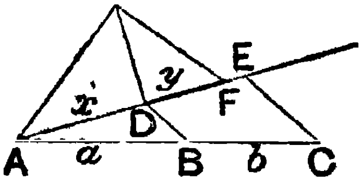
Miscellaneous Problems.

1. Divide the straight line AB at C, so that $AC : AB = 3 : 5$.

2. **Given three straight lines a, b and x, find by construction a fourth y, such that $a^2 : b^2 = x : y$.**

Let AE make any angle with AC; and let $AB = a$, $BC = b$ and $AD = x$.

Construction. Join BD; and through C, draw CE ∥ to BD, to cut AE at E. Take DF the third proportional to AD and DE.



Then DF, is the required fourth line y.

Proof. By const., $AD : DE = DE : DF$; thus $DE^2 = AD.DF$.

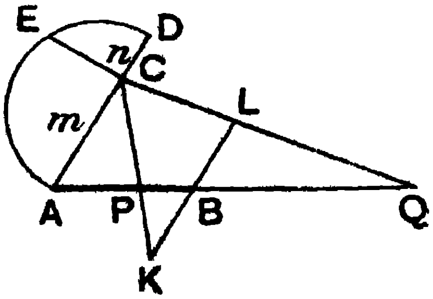
Now $AB^2 : BC^2 = AD^2 : DE^2 = AD : DF$;

that is, $a^2 : b^2 = x : y$.

3. **Divide a given straight line AB (i) internally at P and (ii) externally at Q, such that $AP^2 : PB^2 = AQ^2 : QB^2 = m : n$.**

Construction. Draw AD making any angle with AB. Make $AC = m$ units and $CD = n$ units. Find CE a mean proportional to AC and CD.

Draw KL ∥ to AD; and make $KB = CE = BL$. Join CK, CL; and let CK, CL cut AB at P and Q. Then AB is divided at P and Q as required.



Proof. By similar $\triangle$s **APC, BPK,**

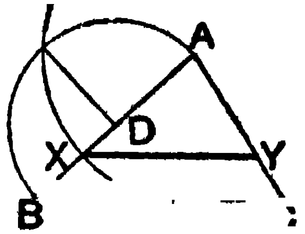
$$\frac{AP^2}{PB^2}=\frac{AC^2}{BK^2}=\frac{AC^2}{CE^2}=\frac{AC}{CD}=\frac{m}{n}.$$

And by similar $\triangle$s **AQC, BQL,**

$$\frac{AQ^2}{QB^2}=\frac{AC^2}{BL^2}=\frac{AC^2}{CE^2}=\frac{AC}{CD}=\frac{m}{n}.$$

4. *To bisect a triangle by a straight line parallel to one of the sides.*

Let ABC be a triangle.



Construction. Bisect AB at D. Find AX the mean proportional to AD and AB. Through X, draw XY parl. to BC.

Then XY bisects the $\triangle$ ABC.

Proof. Because the $\triangle^s$ AXY, ABC are similar ;

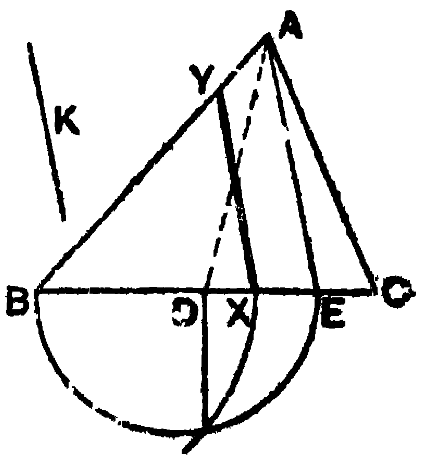
$$\therefore \frac{\triangle AXY}{\triangle ABC}=\frac{AX^2}{AB^2}=\frac{AD.AB}{AB^2}=\frac{AD}{AB}=\frac{1}{2};$$

that is, the $\triangle AXY=\frac{1}{2}\triangle$ ABC.

5. Trisect a triangle by straight lines parallel to one of the sides.

[*Hint.* Trisect **AB** at **D** and **E**. Find mean proportionals **AX** to **AD, AB** ; and **AY,** to **AE, AB.**]

6. *To bisect a triangle by a straight line parallel to a straight line given in position.*



Let ABC be a triangle, and K a given st. line.

Construction. Bisect BC at D. Draw AE parl. to K meeting BC in E. Take BX the mean proportional to BD and BE. Draw XY parl. to AE.

Then XY bisects the $\triangle$ ABC.

Proof. Join AD.

Then because $\triangle^s$ YBX, ABE are similar;

$$\therefore \frac{\triangle YBX}{\triangle ABE} = \frac{BX^2}{BE^2} = \frac{BD \cdot BE}{BE^2} = \frac{BD}{BE} = \frac{\triangle ABD}{\triangle ABE};$$

$$\therefore \triangle YBX = \triangle ABD = \tfrac{1}{2} \triangle ABC.$$

7. Bisect a triangle by a straight line perpendicular to one of the sides.

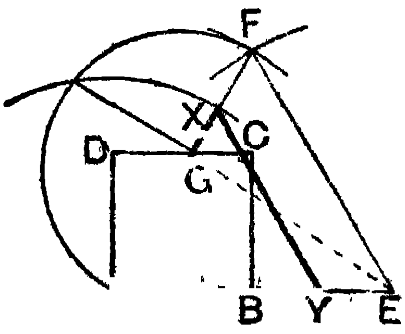
[*Hint.* Bisect **BC** at **D**, and draw **AE** perp. to **BC.**]

8. *To construct an equilateral triangle equal to a given square.*

Let ABCD be the given square.

Construction. Produce AB to E, making BE = AB. On AE, construct an equilateral $\triangle$ AEF. Let AF cut CD in G. Find AX the mean proportional to AG and AF. Draw XY ∥ to FE to meet AE in Y.

Then AXY is the required triangle.



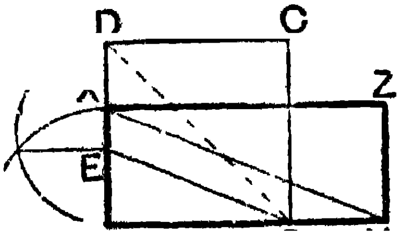
[Supply proof. Observe sq. ABCD = $\triangle$ AGE.]

8. (i) Construct an equilateral triangle equal to a given triangle.

9. *To construct a rectangle equal to a given square and having its adjacent sides in a given ratio.*

Let ABCD be the given square.

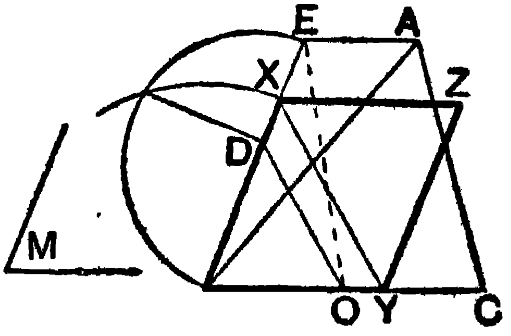
Construction. Make AB : AE = given ratio; and find AX, the mean proportional to AE and AD. Join EB; and draw XY ∥ to EB, meeting AB produced in Y. Complete the rectangle AXZY.



Then AXZY is the rectangle required.

[Supply proof.]

10. *To construct a parallelogram, equal to a given triangle and having an angle equal to a given angle and adjacent sides in a given ratio.*



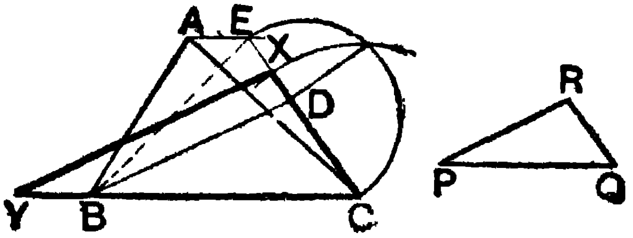
Let ABC be the given triangle, and **M** the given angle.

Construction. Bisect BC at O. Make the $\angle$ OBD = the $\angle$M ; and OB : BD = given ratio. Draw AE parallel to BC, meeting BD produced in E. Find BX the mean proportional to BD, BE. Join OD ; and draw XY ∥ to OD meeting BC in Y. Complete the parm. BXZY.

Then BXZY is the required parallelogram.

[Supply proof.]

11. *To construct a triangle similar to one given triangle and equal in area to another given triangle.*



Let ABC and PQR be the two given triangles.

Construction. Make the $\angle$CBD = the $\angle$P and the $\angle$BCD = the $\angle$Q. Through A, draw AE parl. to BC meeting CD in E. Find CX the mean proportional to CD and CE. Draw XY parl. to DB meeting CB in Y.

Then the $\triangle$ YCX is similar to the $\triangle$ PQR and equal to the $\triangle$ ABC.

Proof. Join BE. Because BD is parl. to XY,
∴ the $\triangle$ YCX is simlr. to the $\triangle$ BCD.

$$\text{Thus} \quad \frac{\triangle YCX}{\triangle BCD} = \frac{CX^2}{CD^2} = \frac{CE}{CD} = \frac{\triangle EBC}{\triangle BCD},$$

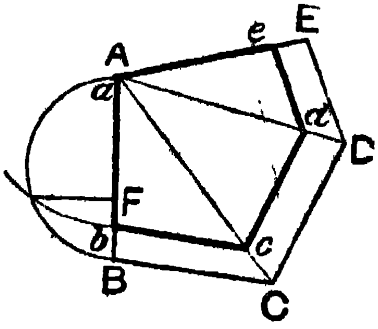
∴ the $\triangle$YCX = the $\triangle$ EBC = the $\triangle$ ABC.

And the $\angle Y$ = the $\angle CBD$ = the $\angle P$; and the $\angle YCX$ = the $\angle Q$;

$\therefore$ the $\triangle$ YCX is similar to the $\triangle$ PQR.

12. *To construct a polygon similar to a given polygon and equal in area to a given fraction* (say *two-thirds*) *of it.*

Let ABCDE be a polygon.



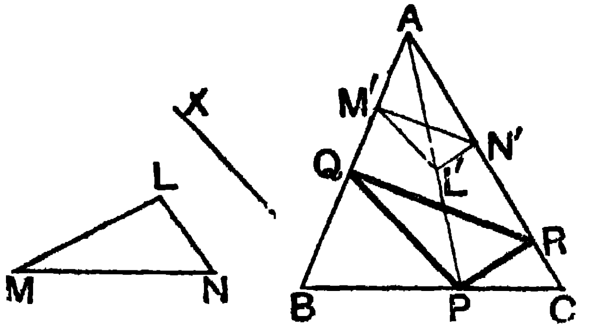
Construction. Divide AB at F, so that AF : AB = 2 : 3. Take A*b* the mean proportional to AF and AB. Join AC, AD. Draw *bc*, *cd*, *de* parl. to BC, CD, DE cutting AC, AD, AE in *c*, *d*, *e*.

Then A*bcde* is simlr. to ABCDE and equal to $\frac{2}{3}$ of it.

[Supply proof. Observe that A*b* and AB are a pair of corresponding sides of two simlr. polygons.]

Note. Similarly a polygon may be constructed similar to a given polygon and having a given ratio to it.

13. *In a given triangle, to inscribe a triangle similar to a given triangle and having a side parallel to a given straight line.*



Let ABC, LMN be two given $\triangle^s$, and X a given st. line.

Construction. Draw any st. line M′L′ ‖ to X. Make the $\angle$L′M′N′ = the $\angle$M; and the $\angle$M′N′L′ = the $\angle$N. Let M′L′ and N′L′ meet at L′. Join AL′; and produce it to meet BC in P.

Draw PQ, PR ‖ to L′M′ and L′N′ respectively. Join QR.

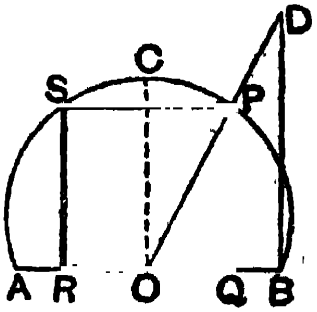
Then the $\triangle$ PQR is simlr. to the $\triangle$ LMN and has the side PQ ‖ to X.

[Supply proof.]

14. Inscribe an equilateral triangle in a given triangle, having a side parallel to the base.

15.* *Inscribe a square in a given segment of a circle, so that two of its vertices are on the arc and the other two on the base of the segment.*

Let ACB be a segment of a circle of which the base is AB; and let O be the middle point of AB.



Construction. Draw BD perp. to AB and equal to AB. Join OD, cutting the arc at P. Draw PQ perp., and PS parl., to AB. Make OR = OQ and join SR.

Then PQRS is the square inscribed in ACB.

Proof. Draw OC perp. to AB. Then OC passes through the centre and bisects PS, for AB and PS are parallel chords. Now OP is a parm.; ∴ OQ = the opp. side = $\frac{1}{2}$ PS.

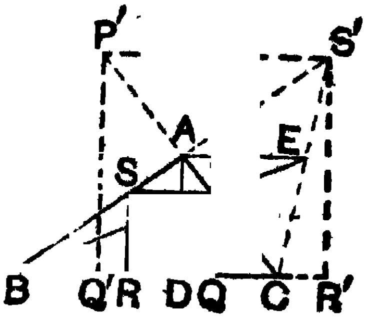
Hence PS = QR; and they are parallel;
∴ PQRS is a parm.

Again by simlr. Δs OQP, OBD,
PQ : QO = DB : BO = 2 : 1; ∴ PQ = 2OQ = QR,
And the ∠ PQR is a right angle.
∴ PQRS is a square.

16. Inscribe a square in a given semi-circle, so that two of its vertices are on the arc and the other two on the diameter of the semi-circle.

17. *Inscribe a square in a given triangle, so that two of its vertices are on two of the sides and the other two on the third side.*

Let ABC be a triangle.



Construction. Draw AD perp., and AE parl. to BC. Make AE = AD. Join BE, cutting AC in P. Draw PQ perp., and PS parl., to BC. Let SR be perp. to BC.

Then PQRS is the square inscribed in the Δ ABC.

Proof. By construction, **PQRS** is a rectangle.

And PQ : AD = CP : CA = BS : BA = SP : AE ;

∴ PQ = PS, for AD = AE (*const.*)

Hence **PQRS** is a square.

Note. This Problem is a particular case of the following Problem—

To construct a square, so that two of its vertices are on the sides (or the sides produced) of a given triangle and the other two on the base.

There are *two* cases. The first is that in which two of the vertices of the square are *on the two sides* as in Prob. 17; and the second is that in which those vertices are *on the sides produced.*

The construction for this case is—Draw **AD** and **AE** as before. Join **CE**, and produce it to meet **BA** produced in **S′**. Draw **S′P′** parl. to **BC** to meet **CA** produced in **P′**. Draw **P′Q′** and **S′R′** perp. to **BC**, produced if necessary. Then **P′Q′R′S′** is the required square.

MISCELLANEOUS EXERCISES LXXXVI.

1. **ABC** is a triangle; through **M** the middle point of **BC**, a line is drawn cutting **AB**, **AC** at **P** and **Q**, and the parallel to **BC** through **A** at **R**. Prove that **PM** : **QM** = **PR** : **QR**.

2. **ABC** is a triangle right-angled at **A**, and **AD** is perpendicular to **BC**. If **BP** bisects the ∠**B** meeting **AD** in **P**, prove that

$$\triangle ABC : \triangle ABD = AP^2 : PD^2.$$

3. **ABCD** is a cyclic quadrilateral; **BL**, **DM** are perpendiculars to **AC**; prove that **BL** : **DM** = **AB.BC** : **AD.DC**.

4. The bisector of the ∠**A** of a triangle **ABC** meets **BC** in **D** and the perpendicular bisector of **BC** in **P**; prove that $PB^2 = PA.PD$.

5. **ABC** is a triangle inscribed in a circle; **AD** is drawn parallel to the tangent at **B** to meet **BC** in **D**; shew that **AD** is a fourth proportional to **BC**, **AB** and **AC**.

6. **OA**, **OB** are tangents to a circle, and **OCD** is a secant cutting the circle at **C**, **D**; prove that in the quadrilateral **ACBD**, the rectangle contained by one pair of opposite sides is equal to the rectangle contained by the other pair.

7. **APB** is a semi-circle whose centre is **O** and diameter **AB** ; **OP**, perpendicular to **AB**, meets the circumference at **P** ; any chord **AQ** cuts **OP**, or **OP** produced, at **R** ; shew that **AP** is a tangent to the circum-circle of the triangle **PQR**.

8. **AB** is a diameter of a circle ; from a point **Q** in **AB** produced, a tangent **QT** is drawn to the circle ; if **TP** is perpendicular to **AB**, then **P** and **Q** divide **AB** internally and externally in the same ratio.

9. The diagonal **AC** of a quadrilateral **ABCD** bisects the $\angle$**A**, and **HK** is drawn parallel to the other diagonal **BD**, cutting **AC** in **E**, **CB** in **H**, **CD** in **K** and **AB**, **AD** in **F** and **G** ; prove that

$$EH : EK = AF : AG.$$

10. **ABCD** is a quadrilateral ; **AF** parallel to **BC** meets **BD**, or **BD** produced, in **F** ; and **BE** parallel to **AD** meets **AC**, or **AC** produced, in **E** ; shew that **EF** is parallel to **CD**.

[*Hint.* If **O** is the intersection of diagonals, prove **OC** : **OE**= **OD** : **OF**.]

11. **ABC** is a triangle and **P** any point in **AB** produced. If **Q** is a point in **BC** such that **CQ** : **BQ**=**AP** : **BP**, prove that **PQ** produced bisects **AC**.

[*Hint.* Draw **BR** || to **AC**.]

12. **ACB** is a semi-circle whose diameter is **AB** ; **BD** is the tangent at **B** and cuts **AC** in **D** ; the $\angle$**BDC** is bisected by **DE** meeting **BC** in **E** ; **EF** is parallel to **AB** meeting **AC** in **F**. Shew that **BF** produced bisects the arc **AC**.

13. From the vertex of a triangle, draw a line to meet the base produced, so that it may be a mean proportional between the segments of the base.

14. **ABC** is a triangle in which **AD**, **BE** are perpendiculars to **BC**, **CA**. If **EP** is perpendicular to **BC**, prove that

$$\triangle ABC : \triangle DEC = BC : CP.$$

15. **ABC** is a triangle inscribed in a circle and **AD**, a tangent to the circle, meets **BC** produced in **D**. Prove that the diameters of the circles about **ABD** and **ACD** are as **AD** : **CD**.

16. **AB** is a diameter of a circle, **CD** a chord parallel to it, and **AC**, **AD** produced cut the tangent at **B** in **E**, **F** respectively. Shew that

$$AB^2 = AC.CE + AD.DF$$

17. **ABC** is a triangle ; **AB**, **AC** are produced to any points **D**, **E**, so that **DE** is parallel to **BC** ; **DE** is divided at **F**, so that **DF** : **FE**=**BD** : **CE**. Prove that the locus of **F** is a straight line.

18. **ABC** is a triangle inscribed in a circle ; the perpendicular **AD** to **BC** meets the circle at **E** ; **AF** parallel to **CE** meets **CB** in **F** ; shew that

$$AB.AC=AF.BE$$

[*Hint.* Triangles **FAC, ABE** are similar.]

19. **ABC** is a triangle ; **AD** bisects the $\angle A$ and meets **BC** in **D** ; **EDF** is drawn perpendicular to **AD** and meets **AB** in **E, AC** in **F.** Prove that

$$AB : AC=BE : FC.$$

[*Hint.* Through **F,** draw **FG** parallel to **AB** meeting **BC** in **G** ; then prove **BE=FG.**]

20. **PQ** is a chord of a circle parallel to the diameter **AB** ; **AQ, BP** intersect at **O** within the circle. Find a point **X** in **AB,** such that

$$AB : AX=\triangle AOB : \triangle POQ.$$

21. **ABC** is a triangle right-angled at **A** ; **AD** is drawn perpendicular to **BC** ; **BE, AF** bisect the $\angle$s **ABD** and **DAC** respectively, meeting **AD** in **E** and **CD** in **F** ; prove that **EF** is parallel to **AC.**

22. **ABCD** is a prallelogram ; any st. line **APQR** cuts **BD** in **P, CD** in **Q** and **BC** produced in **R.** Prove that $PQ : PR=PD^2 : PB^2$, and that **AP** is a mean proportional between **PQ** and **PR.**

23. **OT** is a tangent to a circle and **OPQ** is a secant. If **M** is the middle point of the arc **PQ** and **MT** cuts the chord **PQ** in **A,** shew that

$$AP^2 : AQ^2=OP : OQ.$$

24. **ABC** is a triangle ; **P, Q** are points in **AB** and **AC** such that **BP=CQ** ; if **PQ, BC** when produced meet in **R,** prove that

$$AB : AC=QR ; PR.$$

[*Hint.* Draw **QT** parallel to **AB** cutting **BC** in **T.**]

25. **ABC** is a triangle right-angled at **A** ; **CD** bisects the $\angle C$ and meets **AB** in **D** ; prove that $AB : AC=BC-AC : AD$.

[*Hint.* Draw **DE** perpendicular to **BC.**]

26. Two circles touch internally at **P** and a chord **AB** of the greater touches the less at **C** ; **PA, PB** cut this circle at **Q** and **R** ; prove that

$$PQ : PR=AC : CB.$$

27. From a point **P,** a tangent **PA** and a secant **PBC** are drawn to a circle. If the bisector of the $\angle APB$ meet **AB** in **H** and **AC** in **K,** shew that $AH^2=HB.KC$.

[*Hint.* Prove that **AH=AK.**]

28. **ABCD** is a quadrilateral inscribed in a circle. Find a point **P** in the circumference such that the rect. **PA,PC**=the rect. **PB,PD**.

How many such points can be found ?

29. **O** is any point in the median **AX** of a triangle **ABC**, and **CO**, **BO** meet **AB, AC** in **P** and **Q**. Prove that **PQ** is parallel to **BC**.

30. **ABC** is a triangle ; st. lines **AD, AE** are drawn to the base **BC** making equal angles with **AB, AC** respectively ; prove that

$$AB^2 : AC^2 = BD.BE : CD.CE.$$

31. **ABC** is a triangle inscribed in a circle ; **AD, AE** are lines drawn to the base **BC** parallel to the tangents at **B, C** respectively ; prove that

$$BD : CE = AB^2 : AC^2.$$

[*Hint.* ∆s **DBA, EAC** are similar.]

32. **AD** is the bisector of the ∠**A** of a triangle **ABC** ; **AL, AM** are drawn making equal angles with **AD, AL** meeting **BC** in **L** and **AM** meeting the circum-circle in **M**. Shew that **AL.AM** is constant.

33. **AP** is a tangent from an external point **A** to a circle of centre **B** ; **PQ** parallel to **AB** meets the circle again in **Q** ; prove that

$$AB.PQ = 2\ (\text{radius})^2.$$

34. **AB** is a diameter of a circle and **C** any point on the circumference. If the perpendicular to **AB** at any point **O** cuts the circle in **X** and **AC, BC** in **P, Q** ; prove that $OX^2 = OP.OQ$.

[*Hint.* ∆s **AOP, QOB** are similar.]

35. From the extremities of any chord **AB** of a circle, perpendiculars **AQ, BR** are drawn to the tangent to the circle at any point **P**. If **PM** is perpendicular to **AB**, prove that $PM^2 = AQ.BR$.

36. **ABCD** is a parallelogram and **XY** any straight line cutting **AB, AD**, produced, if necessary, at **P** and **Q** ; **PD** meets **BC** at **M** and **QB** meets **CD** at **N** (**BC** and **CD** being produced if necessary) ; prove that **MN** is parallel to **XY**.

37. Given the vertical angle, the ratio of the sides containing it, and the diameter of the circumcircle, to construct the triangle.

38. **AB** is a chord of a circle of centre **O**, and **C** is any point on the circumference. If the perpendicular bisector of **AB** cuts **AC, BC** in **P** and **Q**, prove that $OP.OQ = (\text{radius})^2$.

[*Hint.* Draw the diameter **COD**, and join **AD**. Prove ∠**OCP**= ∠**OQC** and ∆s **OCP,OQC** are similar.]

39. If **LMN** is the pedal line of a point **P** with respect to a triangle **ABC** [Fig. p. 277], shew that

$$PA.PL = PB.PN = PC.PM.$$

40. **ABC** is a triangle; equal st. lines **AX, BY, CZ** are drawn to meet the opposite sides in **X, Y** and **Z**; through any point **O** within the triangle, **OP, OQ, OR** are drawn parallel respectively to **AX, BY** and **CZ** to meet the opposite sides in **P, Q, R.** Prove that

$$AX = OP + OQ + OR.$$

[*Hint.* $\triangle OBC : \triangle ABC = \text{alt. of } \triangle OBC : \text{alt. of } \triangle ABC$
$= OP : AX$; &c.]

41. If a, b, c are the sides of a triangle **ABC**, and **X, X'** the points where the internal and external bisectors of $\angle A$ meet **BC**. prove that

$$XX' = \frac{2abc}{b^2 - c^2}.$$

Find the values of **XX'** corresponding to the angles **B** and **C**.

42. If a, b, c are the sides of a triangle **ABC**, and **X, X'** the points where the internal and external bisectors of $\angle A$ meet **BC**, prove that

$$\text{(i) } AX^2 = bc\left\{1 - \frac{a^2}{(b+c)^2}\right\};$$

$$\text{(ii) } AX'^2 = bc\left\{\frac{a^2}{(b-c)^2} - 1\right\}.$$

Hence find the value of **XX'** in terms of a, b, c.

43. **ABCD** is a cyclic quadrilateral; **BA, CD** intersect in **P**, and **AC, BD** in **Q**; if **PQ** cuts **AD** in **R**, prove that

$$AR : DR = PA.AB : PD.DC.$$

44. **A, B** are two fixed points and **P** a moveable point on a given circle; in **PB** a point **D** is taken such that the ratio **PD : PA** is constant; in **PA** a point **E** is such that **PE** is to **PB** in the same ratio; prove that **DE** always touches a fixed circle.

45. Tangents **AB, AC** are drawn to a circle; **CE** is perpendicular to the diameter **BD** through **B**; prove that **AD** bisects **CE**.

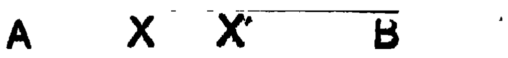
46. If two equal circles touch at **O** and a tangent at **P** to one of them cut the other in **Q** and **R**, prove that **OP** is a mean proportional between **OQ** and **OR**.

47. Equilateral triangles are described on the sides of a right-angled triangle. Prove that the area of the triangle on the hypotenuse is equal to the sum of the areas of the other two triangles.

APPENDIX V

I. SOME USEFUL THEOREMS

I. Theorem. *A finite straight line can be divided in a given ratio at one, and only one, point, whether it be divided internally or externally.*



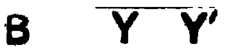
If possible let AB be divided *internally* at two points X, X′ in the ratio $m : n$.

Proof. Then $\frac{AX}{XB} = \frac{AX'}{X'B}$, for each $= \frac{m}{n}$

$$\therefore \frac{AX}{AX+XB} = \frac{AX'}{AX'+X'B}; \qquad [\textit{Lem. } 12.$$

$$\therefore \frac{AX}{AB} = \frac{AX'}{AB}; \text{ i.e., } AX = AX'.$$

Thus X′ coincides with X.



Again, if possible, let AB be divided *externally* at two points Y, Y′ in the ratio $m : n$.

Proof. Then $\frac{AY}{YB} = \frac{AY'}{Y'B}$, for each $= \frac{m}{n}$

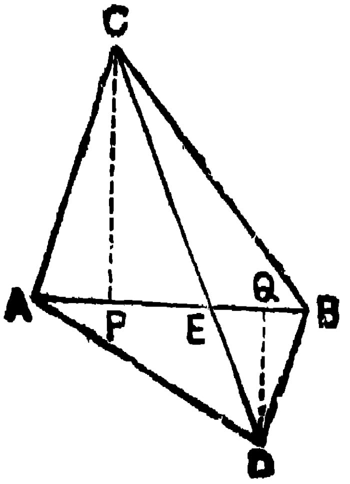
$$\therefore \frac{AY}{AY-YB} = \frac{AY'}{AY'-Y'B}; \qquad [\textit{Lem. } 12.$$

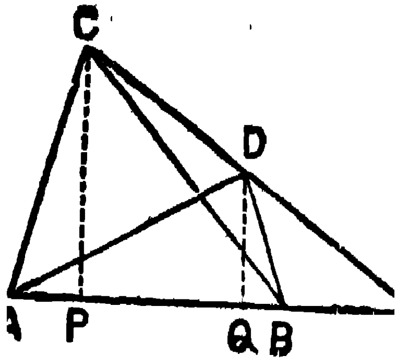
$$\therefore \frac{AY}{AB} = \frac{AY'}{AB}; \text{ i.e., } AY = AY'.$$

Hence Y′ coincides with Y.

Q. E. D.

2. THOEREM. *Triangles on the same base have the same ratio as the segments into which the join of their vertices is divided by the base.*





Let ABC, ABD be two triangles on the same base AB; and let AB cut CD at E.

To prove that $\triangle ABC : \triangle ABD = CE : ED$.

Draw CP, DQ perpendicular to AB.

Proof. Because triangles on the same base have the same ratio as their altitudes; [V. 1, *Cor.* 2.

$$\therefore \triangle ABC : \triangle ABD = CP : DQ$$

$$= CE : ED,$$

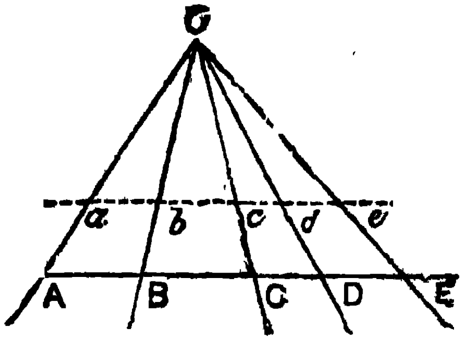
from simlr. $\triangle^s$ CPE, DQE;

that is, $\triangle ABC : \triangle ABD = CE : ED$.

Q. E. D.

Def. Any straight line drawn across a system of straight lines is called a **transversal**; as ABCDE in Fig. of (3).

3. **THEOREM.** *The intercepts made on any transversal by a system of concurrent straight lines are proportional to the corresponding intercepts made on any parallel transversal.*



Let ABCDE be a transversal, cutting a system of concurrent st. lines OA, OB, OC, OD, OE; and let *abcde* be any other transversal parallel to it.

To prove that $\dfrac{AB}{ab}=\dfrac{BC}{bc}=\dfrac{CD}{cd}=\dfrac{DE}{de}$.

Proof. Because ABCDE is ‖ to *abcde*;

∴ the △ OBA is simlr. to the △ O*ba* and the △ OBC to the △ O*bc*;

$$\therefore \frac{AB}{ab}=\frac{OB}{Ob}=\frac{BC}{bc}\text{; that is, }\frac{AB}{ab}=\frac{BC}{bc}.$$

$$\text{Similarly } \frac{BC}{bc}=\frac{OC}{Oc}=\frac{CD}{cd}\text{; that is, }\frac{BC}{bc}=\frac{CD}{cd}.$$

And so on.

$$\text{Hence}\quad \frac{AB}{ab}=\frac{BC}{bc}=\frac{CD}{cd}=\frac{DE}{de}.$$

Q. E. D.

Cor. In the case of *three* concurrent lines OA, OB, OC, we evidently have AC : BC = *ac* : *bc*.

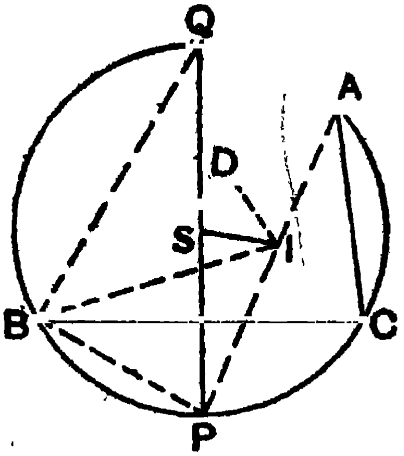
Ex. 1. If three concurrent straight lines OA, OB, OC are cut by two transversals at *a*, *b*, *c* and A, B, C, so that *ab* : *bc* = AB : BC, then these transversals are parallel.

4. THEOREM. *If* S, I *are the centres and* R, r *the radii of the circum-circle and the incircle respectively of a triangle* ABC, *then*

$$SI^2 = R^2 - 2Rr.$$

Const. Join AI, BI; and produce AI to meet the circum-circle at P. Draw the diameter PQ; and join PB, BQ. Draw ID perp. to AB.

Then $PQ = 2R$ and $ID = r$.



Proof. Because the ∠BIP = the ∠IAB + the ∠IBA = the ∠PAC + the ∠IBC = the ∠PBC + the ∠IBC = the ∠IBP;

∴ IP = PB.

Also the ∠IAD = the ∠PQB in the same segment and the ∠s IDA, PBQ are rt. angles;

∴ the △s IAD, PQB are similar;

thus IA : PQ = ID : PB;

∴ IA.PB = PQ.ID.

Now because S is the centre of the circle ABC and AP a chord through I; therefore from IV. 11, *Note*,

$$R^2 - SI^2 = IA.IP = IA.PB = PQ.ID;$$

that is, $R^2 - SI^2 = 2Rr$;

$$\therefore SI^2 = R^2 - 2Rr. \qquad \text{Q. E. D.}$$

5. THEOREM. *If* S, I_1 *are the centres and* R, r_1 *the radii of the circum-circle and the ex-circle touching the side* BC *of a triangle* ABC, *then*

$$SI_1^2 = R^2 + 2Rr_1.$$

[Join AI_1 cutting the circum-circle at P. Draw the diam. PQ. Join I_1B, PB and BQ. Draw I_1D perp. to AB. Then prove that the ∠ I_1BP = the ∠ BI_1P; and hence $I_1P = PB$. Also by simlr. △s I_1AD, PQB, prove that $I_1A.PB = PQ.I_1D$.]

Obs. Similar expressions for the other two ex-circles are

$$SI_2^2 = R^2 + 2Rr_2 \text{ and } SI_3^2 = R^2 + 2Rr_3.$$

II. HARMONIC SECTION

DEFINITIONS

If a straight line is divided internally and externally in the same ratio, it is said to be **divided harmonically.**

Thus if AX : XB = AY : YB, then AB is divided harmonically at X and Y.

The internal and external bisectors of an angle of a triangle divide the opposite side harmonically.

A series of points lying in a straight line is called a **range**. A range of four points A, B, C and D may be written (ABCD). If a range of **four** points A, X, B, Y, is such that AB is divided harmonically at X and Y, it is called a **harmonic range,** and the points X and Y are called **harmonic conjugates** of the points A and B.

A system of straight lines passing through a point is called a **pencil** ; each of the straight lines is called a **ray ;** and the point of concurrence is called the **vertex** of the pencil.

Thus if (AXBY) is a harmonic range and O a pt. without it, then the lines OA, OX, OB and OY, passing through O, form a harmonic pencil, in which OA, OX, OB and OY are rays, and O the vertex.

In a harmonic pencil, the alternate rays OX, OY are called harmonic conjugates of the rays OA, OB.

The notation O(AXBY) is used to denote a pencil whose vertex is O and rays are OA, OX, OB, OY.

6. **Theorem** *If* AB *is divided harmonically at* X *and* Y *then* XY *is divided harmonically at* A *and* B.

A X B Y

Proof. By def., AX : XB = AY : YB ;

∴ alternately, AX : AY = XB : YB ;

that is, YB : BX = YA : AX.

Thus XY is divided harmonically at B and A.

Q. E. D.

This Theorem is otherwise enunciated thus:—*If* X and Y *are harmonic conjugates of* A and B, *then* A and B *are harmonic conjugates of* X and Y.

Def. Three magnitudes are said to be in **Harmonical Progression,** when

first : third = first − second : second − third.

Thus a, b, c are in harmonical progression, when

$$\frac{a}{c}=\frac{a-b}{b-c}.$$

The middle magnitude b is called the **harmonic mean** between a and c.

If a, b, c are in Harmonical Progression, then

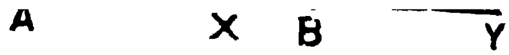
$$\frac{a-b}{b-c}=\frac{a}{c}, \text{ or } \frac{a-b}{b-c}=\frac{ab}{bc},$$

$$\frac{a-b}{ab}=\frac{b-c}{bc}, \text{ i.e., } \frac{1}{b}-\frac{1}{a}=$$

Thus $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in Arithmetical Progression.

Hence *if* a, b, c *are in Harmonical Progression, their reciprocals are in Arithmetical Progression.*

7. THEOREM. *If* AB *is divided harmonically at* X and Y, *then* AB *is a harmonic mean between* AX *and* AY.



Proof. By definition,

$$AY : YB = AX : XB,$$

$$\therefore \text{ alternately, } AY : AX = YB : XB,$$

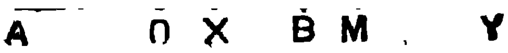
$$\therefore AY : AX = AY - AB : AB - AX.$$

Thus; AY, AB, AX are in harmonical progression; so that AB is the harmonic mean between AX and AY.

Q. E. D.

Note. XY *is a harmonic mean between* YB *and* YA; for XY is divided harmonically at B and A.

8. THEOREM. *If* AB *is divided harmonically at* X *and* Y, *and* O *is the middle point of* AB, *then* $OB^2 = OX.OY$.



Proof. Because O is the middle point of AB, therefore

$$\left.\begin{aligned} AX &= OB + OX \\ XB &= OB - OX \end{aligned}\right\} \text{ and } \left\{\begin{aligned} AY &= OY + OB, \\ YB &= OY - OB. \end{aligned}\right.$$

$$\text{Now } \frac{AX}{XB} = \frac{AY}{YB}; \quad \therefore \frac{OB + OX}{OB - OX} = \frac{OY + OB}{OY - OB};$$

$$\text{that is, } \frac{OB}{OX} = \frac{OY}{OB}; \quad \therefore OB^2 = OX.OY.$$

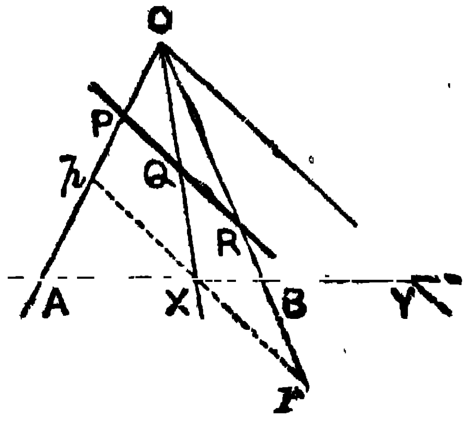
Q. E. D.

Note. If M is the middle point of XY, then* $MX^2 = MB.MA$; for (YBXA) is a harmonic range.

9. THEOREM. *If* O *is the middle point of* AB *and* $OB^2 = OX\,OY$, *then* AB *is divided harmonically at* X *and* Y. [Converse of (8)].

[The proof is left to the student.]

10. THEOREM. *If a transversal is parallel to one of the rays of a harmonic pencil, the other three rays intercept equal portions on it.*



Let a transversal PQR be ∥ to OY, one of the rays of harmonic pencil O (AXBY).

To prove that $PQ = QR$.

Through X, draw pXr ∥ to PQR or OY.

Proof. Then $PQ : QR = pX : Xr$. [*App.* V. (3).

Now by simlr. Δ^s AXp, AYO, $AX : AY = pX : OY$;

and by simlr. Δ^s BrX, BOY, $XB : BY = Xr : OY$.

Also because (AXBY) is a harmonic range.

$\because AX : AY = XB : BY$.

$\therefore pX : OY = Xr : OY$, that is, $pX = Xr$.

But $PQ : QR = pX : Xr$: [*Proved.*

$PQ = QR$. Q. E. D.

11. THEOREM. *In a pencil of four rays if three of the rays intercept equal portions on a transversal parallel to the fourth ray, the pencil is harmonic.* [Converse of (10)].

Let O(AXBY) be a pencil of four rays [see Fig. of (10)]; and let three of the rays OA, OX, OB intercept equal portions PQ, QR on a transversal PQR parallel to OY.

To prove that $O(AXBY)$ is a harmonic pencil.

Through X, draw pXr parallel to PQR or OY.

Proof. Because pXr is parl. to PQR;

$\therefore pX : Xr = PQ : QR = 1$ [*Hyp.*]; $\therefore pX = Xr$.

Now the $\triangle^s$ AXp, AYO are simlr.; $\therefore AX : AY = pX : OY$; also the $\triangle^s$ BrX, BOY are simlr.; $\therefore XB : BY = Xr : OY$.

But $pX = Xr$; [*Proved.*

$\therefore AX : AY = XB : BY$;

$\therefore$ alternately, $AX : XB = AY : BY$.

$\therefore O(AXBY)$ is a harmonic pencil. Q. E. D.

12. THEOREM. *If three of the rays of a harmonic pencil intercept equal portions on a transversal, the transversal is parallel to the fourth ray.* [Converse of (10).]

Let $O(AXBY)$ be a harmonic pencil [see Fig. of (10)]; and let three of its rays OA, OX, OB intercept equal portions PQ, QR on a transversal PQR.

To prove that PQR is parallel to OY.

Proof. If PQR be not parallel to OY, draw OY' parallel to PQR meeting $AXBY$ in Y'.

Then because $PQ = QR$ and PQR parallel to OY', $O(AXBY')$ is a harmonic pencil; [*App.* V. (11).

$\therefore AX : XB = AY' : BY'$.

But by *Hyp.*, $AX : XB = AY : BY$.

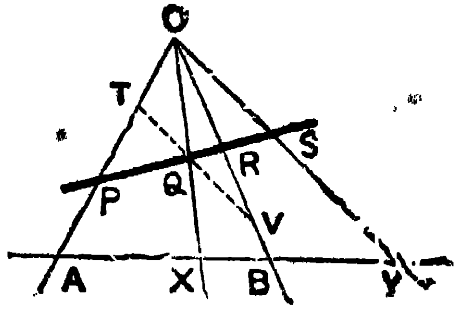
$\therefore AY' : BY' = AY : BY$.

Thus AB is divided externally in the same ratio at two points Y, Y' which is absurd. [*App.* V. (1).

Hence Y' coincides with Y; that is, PQR is parl. to OY.

Q. E. D.

13. THEOREM. *Any transversal, that cuts a harmonic pencil, is cut harmonically by the rays of the pencil.*



Let O(AXBY) be a harmonic pencil, and PQRS any transversal that cuts it.

To prove that PQ : QR=PS : RS.

Through Q, draw TQV parallel to OY ;

then TQ=QV. [*App.* V. (10)

Proof. TQ being parallel to OS,

the △PQT is simlr. to the △PSO ;

∴ PQ : PS=TQ : OS ;

also QV being ∥ to OS, the △QVR is simlr. to the △SOR ;

∴ QR : RS=QV : OS=TQ : OS,

∴ PQ : PS=QR : RS ;

∴ alternately, PQ : QR=PS : RS.

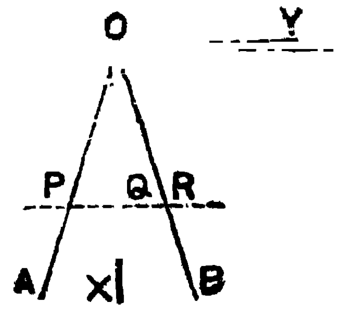
Q. E. D.

14. THEOREM. *In a harmonic pencil, if one of a pair of conjugate rays, bisects the angle between the other pair, it is perpendicular to its conjugate.*

Let O(AXBY) be a harmonic pencil, in which OX bisects the ∠AOB.

To prove that OX is perpendicular to OY.

Take any pt. **P** in **OA**; draw **PQ** perp. to **OX**, and produce it to cut **OB** in **R**.



Proof. The ∠**POQ**=the ∠**ROQ**, and the ∠s **OQP**, **OQR** are rt. angles;

∴ △s **OPQ**, **ORQ** are congruent; so that **PQ**=**QR**.

Then ∵ **PQ**=**QR**, ∴ **PQR** is ∥ to **OY**; [*App.* V. (12).

∴ the ∠**YOQ**=the alt. ∠**OQP**=a right angle.

Q. E. D.

15. THEOREM. *In a harmonic pencil, if one pair of conjugate rays are at right angles, then they are the internal and external bisectors of the angle between the other pair.* [Converse of (14)].

Let **O(AXBY)** be a harmonic pencil in which **OX** is perp. to **OY**. [see Fig of (14)].

To prove that **OX**, **OY** bisect the ∠ **AOB** internally and externally.

Draw any transversal **PQR** ∥ to **OY**.

Proof. Then **PQ**=**QR**. [*App.* V. (10).

And **PQR** being ∥ to **OY**, the ∠s at **Q** are right angles.

Hence the △s **OQP**, **OQR** are congruent;

∴ the ∠**POQ**=the ∠**ROQ**.

∴ **OQ** or **OX** bisects the ∠ **AOB** internally.

And ∵ **OX** bisects the ∠**AOB**, and the ∠**XOY** is rt. ∠.

∴ **OY** bisects the ∠ **AOB** externally. Q. E. D.

Ex. 2. The arms of an angle and its internal and external bisectors form a harmonic pencil.

Ex. 3. If **(AXBY)**, is a harmonic range, prove that

$$\frac{2}{\mathbf{AB}}=\frac{1}{\mathbf{AX}}+\frac{1}{\mathbf{AY}}\text{ and }\frac{2}{\mathbf{XY}}=\frac{1}{\mathbf{BY}}+\frac{1}{\mathbf{AY}}.$$

Ex. 4. If **X** and **Y** are harmonic conjugates of **A** and **B** and **O** the middle point of **AB** and **M** that of **XY**, prove that $\mathbf{OB}^2+\mathbf{MX}^2=\mathbf{OM}^2$.

Ex. 5. Any transversal through the middle point of the base of a triangle is cut harmonically by the sides and the line through the vertex parallel to the base.

Ex. 6. Any transversal is cut harmonically by the sides of a triangle, the median to the base and the parallel to the base through the vertex.

Ex. 7. The diameter of a circle perpendicular to a side of an inscribed triangle is cut harmonically by the other two sides.

Ex. 8. If any two diameters of a circle are cut harmonically at **P**, **Q** and **P′**, **Q′**, then **P**, **Q**, **P′**, **Q′** are concyclic.

Ex. 9 A diameter of a circle is cut harmonically by any tangent and the projection of the point of contact on the diameter.

Ex. 10. If *a*, *b*, *c* are the middle points of **BC**, **CA**, and **AB** of a triangle **ABC**, prove that *ac*, *ab* are harmonic conjugates of *a***A**, *a***C**.

Ex. 11. Through the intersection of the diagonals of a parallelogram, straight lines are drawn parallel to the sides. Prove that these lines and the diagonals form a harmonic pencil.

Ex. 12. **(ABCD)**, (*abcd*) are two harmonic ranges, one in each of two straight lines. If any three of the straight lines **A***a*, **B***b*, **C***c*, **D***d*, meet at a point **O**, then shall the fourth pass through **O**.

Ex. 13. **(ABCD)**, **(A***bcd***)** are two harmonic ranges in the arms of an angle **A**. Prove that **B***b*, **C***c*, **D***d* are concurrent.

Ex. 14. If in two harmonic pencils, the three points of intersection of any three pairs of corresponding rays are in a straight line, the point of intersection of the fourth pair shall also be in that line.

Ex. 15. **OA, OD** are tangents to a circle ; **AB** is a diameter and **DC** is perpendicular to **AB** ; prove that **OB** bisects **CD**. [See Ex. 9.]

III. ORTHOGONAL CIRCLES

Def. 1. If two circles intersect, then the angles between their tangents at a point of intersection is said to be *the angle at which the circles intersect.*

Let two circles ABCE and ABDF intersect at A and B; and let AD, AC [Fig. of (16)] be the two tangents at A. Then CAD is the angle at which the circles intersect at A. Similarly if BF, BE are the tangents at B, then EBF is the angle at which the circles intersect at B.

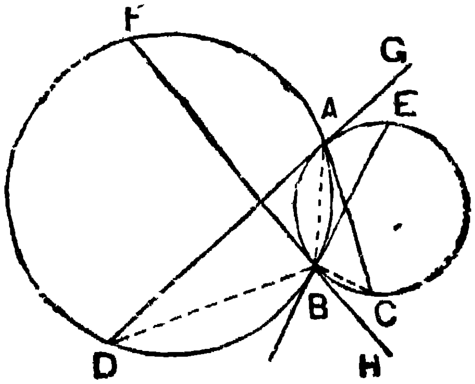
If the ∠CAD = 60°, the circles are said to intersect at an angle of 60°.

Def. 2. Two circles are said to cut **orthogonally** (*i e.*, at right angles), when the angle between their tangents at a point of intersection is a right angle.

Thus if the ∠CAD is a right angle, then the circles ABCE, ABDF are said to cut orthogonally at A.

The circles which cut orthogonally are called **orthogonal circles**.

16. THEOREM. *The angle at which two circles intersect at one of the points of intersection is equal to the angle at which they intersect at the other point.*



Let the $\odot^s$ ABCE, ABDF cut at the ∠CAD at A, and at the ∠EBF at B.

To prove that the ∠CAD = the ∠EBF.

Construction. Join AB, BC, BD.

Proof. Now the ∠CAB = the ∠D in the alt. segment; and the ∠DAB = the ∠C in the alt. segment.

∴ the ∠CAD = the ∠D + the ∠C.

Similarly the ∠EBF = the ∠EBA + the ∠FBA.

= the ∠D + the ∠C.

∴ the ∠CAD = the ∠EBF.

Q. E. D.

Cor. *If one circle cut another orthogonally at one point of intersection, it also cuts it orthogonally at the other point.*

17. THEOREM. *If two circles cut one another orthogonally, the tangent to either circle at a point of intersection passes through the centre of the other.*

Let two ⊙s U and V [see Fig. of (20)] whose centres are O, O′, cut orthogonally at P.

Let PO be the tangent to V at P.

To prove that PO passes through the centre of U. Draw PO′ the tangent to U at P.

Proof. Because U and V cut orthogonally at P, the ∠OPO′ is a right angle.

Hence OP is perp. to the tangent O′P, and therefore passes through the centre of U. [III. 13, *Cor.* 2.

Similarly it may be proved that O′P passes through the centre of V.

Q. E. D.

18. THEOREM. *A circle* U *cuts a circle* V *orthogonally, if the tangent to* V *from the centre of* U *is equal to the radius of* U.

Let **O** and **O′** be the centres of **U** and **V** [Fig. of (20)].

Let **OP** equal to the radius of **U** be a tangent to **V** at **P**.

To prove that **U** cuts **V** orthogonally at **P**.

Proof. Join **O′P**.

Because **OP** is a tangent to **V**;

∴ the ∠**O′PO** is a right angle.

But **OP**=a radius of **U**, ∴ **P** is on the circumference of **U**. Hence **O′P** touches **U** at **P**.

Thus the tangents at **P** include a right angle;

∴ the circle **U** cuts the circle **V** orthogonally. Q. E. D.

Cor. *If from an external point a tangent is drawn to a circle, the circle with that point as centre and the tangent as radius cuts the given circle orthogonally.*

19. THEOREM. *If two circles cut orthogonally, the square on the line joining their centres is equal to the sum of the squares on their radii*; and *conversely.*

Since the ⊙s are orthogonal, the tangent to either at a pt. of intersection passes through the centre of the other [*App.* V. (17)]. Now apply II. 5, to prove the Theorem, and II. 6, to prove its Converse.

Ex. 16. If from an external point **P**, tangents **PQ**, **PR** are drawn to a given circle, prove that the circle described with centre **P** and radius **PQ** or **PR** is orthogonal to the given circle.

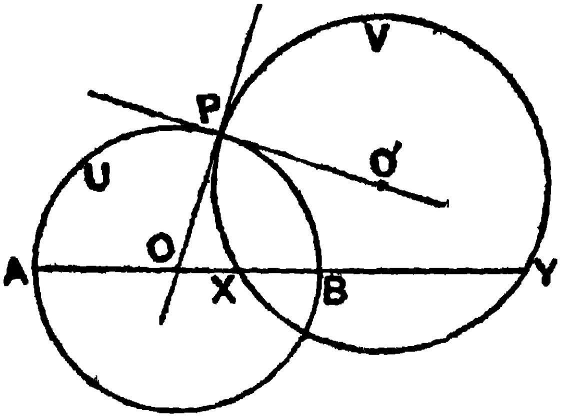
Ex. 17. If from an external point **P** a tangent **PT** is drawn to a given circle, then the circle described on **PT** as diameter cuts the given circle orthogonally.

Shew that another circle, orthogonal to the given circle, can be described passing through **P**.

Ex. 18. Draw a circle with a given centre to cut a given circle orthogonally.

Ex. 19. Draw a circle with a given radius to cut a given circle orthogonally at a given point.

20. THEOREM. *If two circles cut orthogonally, any diameter of one is cut harmonically by the other.*



Let O, O′ be the centres of two circles U and V that cut orthogonally at P ; and let V cut AB, any diameter of U, at X, Y.

To prove that (AXBY) is a harmonic range.

Proof. Because OP is a tangent to V,

$\therefore OX.OY = OP^2$. [IV. 11, *Cor.*

Now the circles U and V are orthogonal ;

∴ the tangent OP to V from O = a radius of U. [*App.* V. (18).

Hence $OX.OY = OP^2 = OB^2$.

And O is the middle pt. of AB.

Thus A, X, B, Y form a harmonic range. [*App.* V. (9).

Q. E. D.

21. THEOREM. *If a diameter of a circle is divided harmonically at* X *and* Y, *any circle through* X *and* Y *cuts the circle orthogonally.* [Converse of (20).]

Let the diameter AB of U [see Fig. of (20)] be cut harmonically at X and Y.

To prove that any circle V through X and Y cuts U orthogonally.

Proof. Because O is the middle pt. of AB, and A, X, B, Y a harmonic range,

$\therefore OB^2 = OX.OY$ [*App.* V. (8).

$=$ the square on the tangent from O to V.

[IV. 11, *Cor.*

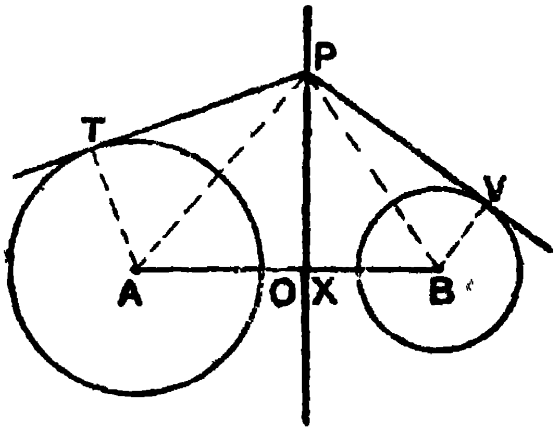
Hence the tangent to V from the centre of U $=$ a radius of U ;

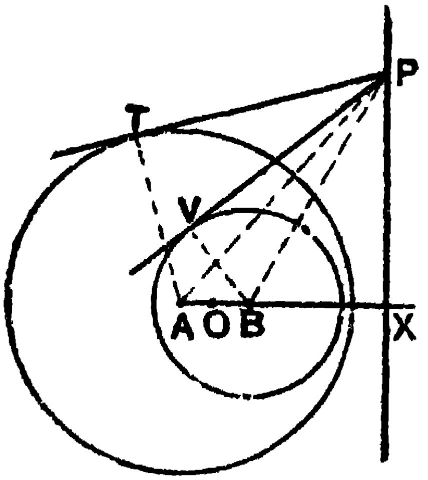
$\therefore$ V cuts U orthogonally. [*App.* IV. (18).

Q. E. D.

IV. RADICAL AXIS

22. THEOREM. *The locus of a point from which tangents drawn to two given circles are equal is a fixed straight line perpendicular to the line of centres.*





Let A and B be the centres, and a and b the radii, of the given circles.

Let P be any point from which the tangent PT drawn to (A) is equal to the tangent PV drawn to (B) ; and let PX be perp. to AB, or AB produced.

To prove that PX is the locus of P.

Construction. Join AP, AT, BP, BV.

Proof. $PT^2 = AP^2 - AT^2 = PX^2 + AX^2 - a^2$;

and $PV^2 = BP^2 - BV^2 = PX^2 + BX^2 - b^2$.

But $PT^2 = PV^2$;

$\therefore PX^2 + AX^2 - a^2 = PX^2 + BX^2 - b^2$;

that is, $AX^2 - a^2 = BX^2 - b^2$.

Hence $AX^2 - BX^2 = a^2 - b^2$.

Now if O be the middle point of AB, then

$$AX^2 - BX^2 = (AX + BX)(AX - BX) = 2OX.AB.$$

$$\therefore 2OX.AB = a^2 - b^2;$$

$\therefore$ X is a fixed point and PX is a fixed st. line.

Thus *every* point P from which tangents to the two circles are equal lies on PX.

Again let P be any point on the perpendicular PX, which cuts AB at X, so that $AX^2 - BX^2 = a^2 - b^2$.

Since $AX^2 - BX^2 = a^2 - b^2$;

$\therefore PA^2 - PB^2 = a^2 - b^2$, [*App.* IV. (3), *Cor.*

$\therefore PA^2 - a^2 = PB^2 - b^2$;

Thus, $PT^2 = PV^2$; that is, $PT = PV$.

Thus *any* point P on PX is such that tangents from it to the given circles are equal.

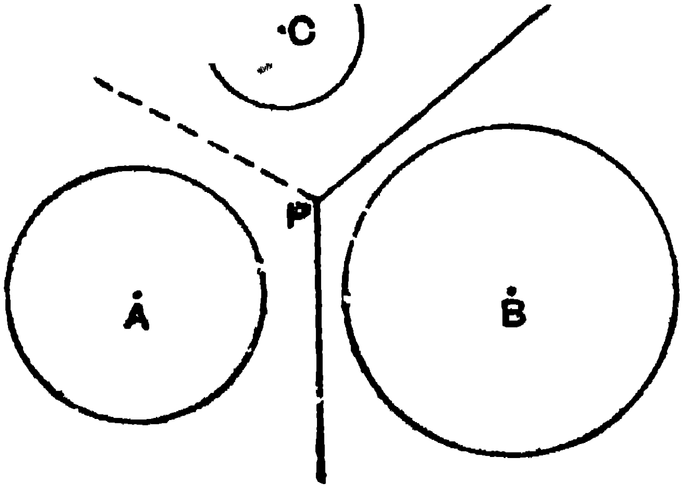
Hence PX is the locus of points from which tangents to the given circles are equal. Q. E. D.

Def. The **radical axis** of two circles is the locus of points from which the tangents to the circles are equal.

Thus PX is the *Radical Axis* of the two circles (A) and (B).

Obs. According to the definition, **the radical axis of two intersecting circles is the common chord produced**, for tangents from any point on it to the circles are equal. But it is usual to regard the **whole line** as the radical axis.

23. THEOREM. *The three radical axes of three circles taken in pairs are concurrent.*



Let the radical axis of (A) and (B) and of (B) and (C) meet at P.

To prove that the radical axis of (A) and (C) passes through P.

Proof. Because P is on the radical axis of (A) and (B),
∴ the tangent from P to (A) = the tangent from P to (B).

For the same reason,

the tangent from P to (B) = the tangent from P to (C).

∴ the tangent from P to (A) = the tangent from P to (C).

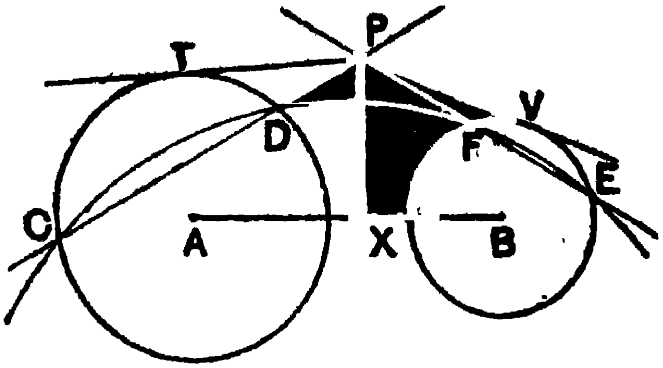
∴ P lies on the radical axis of (A) and (C). Q. E. D.

Cor. *The radical centre of three circles is the centre of the circle that cuts the given circles orthogonally.*

[Use *App.* V. (18), *Cor.*]

Def. The point of concurrence of the radical axes of three circles taken in pairs is called the **radical centre** of the circles.

24. PROBLEM. *To construct the radical axis of two given circles.*



Let (A) and (B) be the given circles.

Construction. Draw any circle cutting the given circles at C, D and E, F.

Join CD, EF, and produce them to meet at P. Draw PX perpendicular to AB, the line of centres.

Then PX is the radical axis required.

Proof. From P, draw the tangents PT, PV to the circles.

$$\text{Then } PT^2 = PC.PD = PE.PF = PV^2 ;$$

$$\therefore PT = PV ;$$

$$\therefore P \text{ is a point on the radical axis ;}$$

that is, PX is the radical axis of (A) and (B).

Ex. 20. What is the radical axis of two circles that touch one another internally or externally ?

Ex. 21. The radical axis of two circles bisects each of the four common tangents.

Ex. 22. If three circles touch one another, the three common tangents at the points of contact are concurrent.

Ex. 23. On the sides of a triangle as diameters, circles are described ; shew that their radical centre is the orthocentre of the triangle. What are their radical axes ?

V. CENTRE OF SIMILARITY

Def. 1. When similar polygons are so placed that their corresponding sides are parallel they are said to be **similarly situated,** or are called **homothetic figures.**

Thus in Figures 1 and 2, the polygons ABCD, *abcd* are similarly placed.

25. THEOREM. *If two similar polygons are similarly placed, the straight lines joining corresponding vertices are concurrent.*

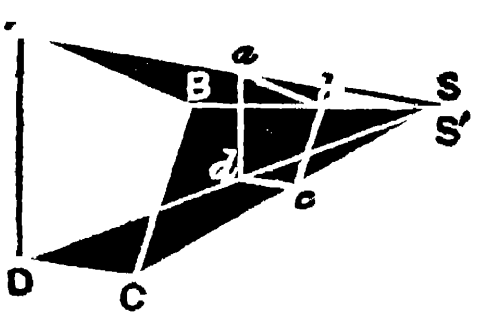


Fig. 1.

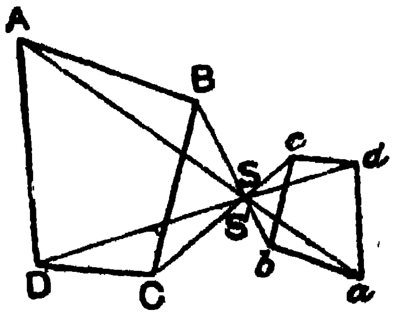


Fig. 2.

Let the similar polygons ABCD, *abcd* be so placed that AB is parallel to *ab*, BC to *bc*, &c.

To prove that A*a*, B*b*, C*c*, &c. are concurrent.

[In Fig. 1, **AB** and *ab*, **BC** and *bc*, &c. are drawn in the *same* sense ; while in Fig. 2, they are drawn in *opposite* senses.]

Let A*a*, B*b*, meet at S. Then C*c* shall pass through S. If not, let it cut B*b* in S.'

Proof. By simlr. $\triangle^s$ ABS, *ab*S,

$$BS : bS = AB : ab$$

$$= BC : bc \qquad [Hyp.$$

$$= BS' : bS'$$

by simlr. $\triangle^s$ BCS', *bc*S';

$$\therefore BS : bS = BS' : bS'.$$

Thus B*b* is divided in the same ratio at S, S′;

∴ S′ coincides with S. [*App*. V. (1).

Hence C*c* passes through S.

Similarly it can be proved that D*d* passes through S.

Q. E. D.

Cor. Hence *the straight line joining corresponding points in two similar polygons is divided at the centre of similarity in a fixed ratio*, viz., *the ratio of the perimeters of the polygons.*

Def. 2. If two similar polygons are similarly situated the point of concurrence of the straight lines joining corresponding vertices is called the **centre of similarity**, or **homothetic centre** of the figures. Thus S is the centre of similarity of ABCD, *abcd.*

VI. CENTRES OF SIMILITUDE

Def. If the straight line joining the centres of two circles is divided internally and externally in the ratio of their radii, either of the points of section is called a **centre of similitude** of the circles.

Thus in the figures of (26), if A*a*, joining the centres of the circles (A) and (*a*) is divided at S, so that

$$SA : Sa = \text{radius of } (A) : \text{radius of } (a),$$

then S is called a centre of similitude of the circles.

The point of internal section is called the **internal centre** of similitude [Fig. 2], and that of external section, the **external centre** of similitude [Fig. 1]. A pair of circles therefore has *two* centres of similitude.

Hence it is obvious that if two circles touch each other, the point of contact is the *internal* centre of similitude when they touch *externally*, and *external* centre of similitude when they touch *internally*.

Note 1. From the definition it is clear that *the centres of similitude of two circles divide the join of the centres* **harmonically.**

Note 2. Since the join of the centres is a fixed line and is divided in a fixed ratio at either centre of similitude, it is evident that *each of the centres of similitude is a* **fixed point.**

26. THEOREM. ***A tangent from a centre of similitude to either circle touches the other.***

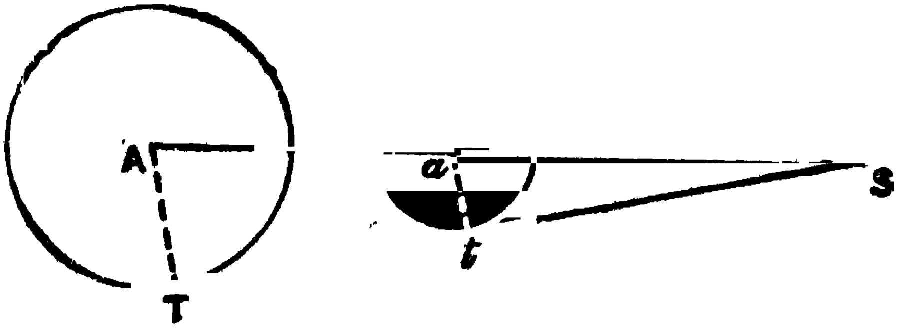


Fig. 1

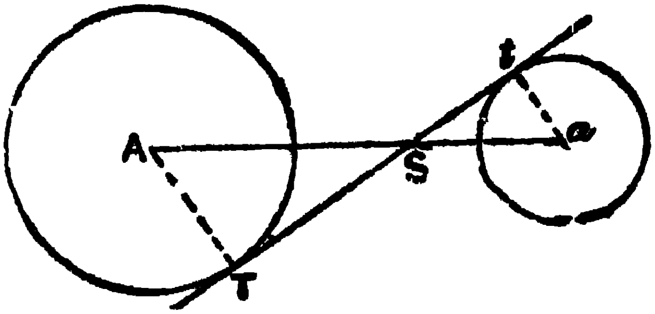


Fig. 2

Let **ST** be a tangent to (A) from **S** a centre of similitude of the circles (A) and (*a*).

To prove that ST touches the circle (*a*).

Join AT, and draw *at* perp. to ST or ST produced.

Proof. Then the Δ^s SAT and S*at* are equiangular ;

thus AT : *at*=SA : S*a*=radius of (A) : radius of (*a*) ;

∴ *at* is a radius of the circle (*a*).

Hence *t* is on the circle (*a*) ; *i.e.*, ST touches (*a*) at *t*.

Q. E. D.

27. THEOREM. *The straight line joining the extremities of any two parallel radii of two circles passes through a centre of similitude.*

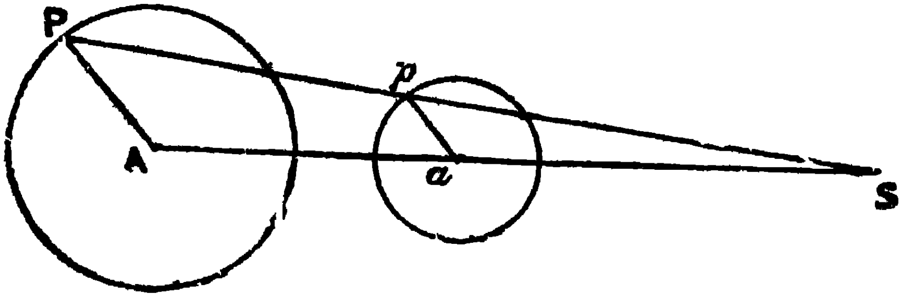


Fig. 1.

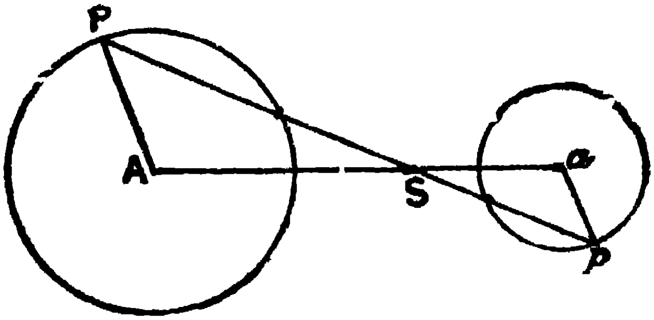


Fig. 2.

Let (A), (a) be two circles whose centres are A and a

Let AP, ap be any two parallel radii, drawn in the *same sense* in Fig. 1 and in *opposite senses* in Fig 2.

To prove that Pp passes through a centre of similitude.

Let Pp cut Aa at S.

Proof. Because AP is parallel to ap ;

∴ the Δ^s SAP, Sap are equiangular ;

∴ SA : Sa=AP : ap=radius of (A) : radius of (a).

Thus Aa is divided at S *externally* in Fig. 1 or *internally* in Fig. 2, in the ratio of the radii.

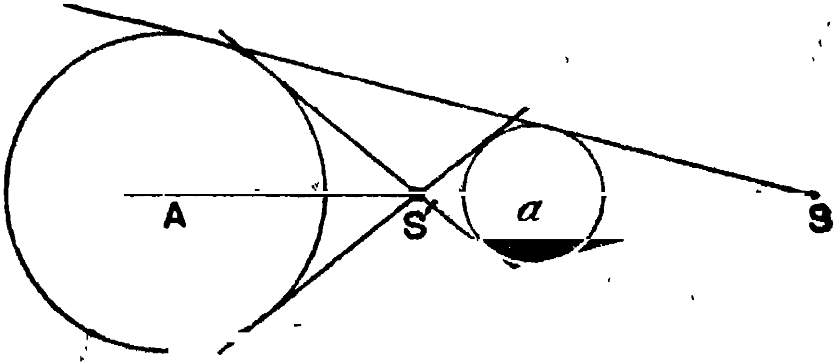
Hence S is a centre of similitude. Q. E. D.

Note that **AP** and ap may have *any* directions ; also **S** is **external,** when the radii are drawn in the **same** sense and **internal** when they are drawn in **opposite** senses.

Cor. 1. *A common tangent to two circles passes through centre of similitude.*

For the radii to the points of contact, being perpendicular to the tangent, are parallel.

Cor. 2. *The pairs of common tangents to two circles intersect on the line of centres at the centres of similitude.*



Obs. From this Theorem, it is easy to see that a **centre of similitude** of two circles is the **point** where the **line of centres** is cut—

(i) by a **common tangent**.

or (ii) by the join of the ends of any two **parallel radii**.

It is **external** or **internal** according as the common tangent is **direct** or **transverse**, or the parallel radii are drawn in the **same** or **opposite** senses.

Question. How many centres of similitude have two equal circles ?

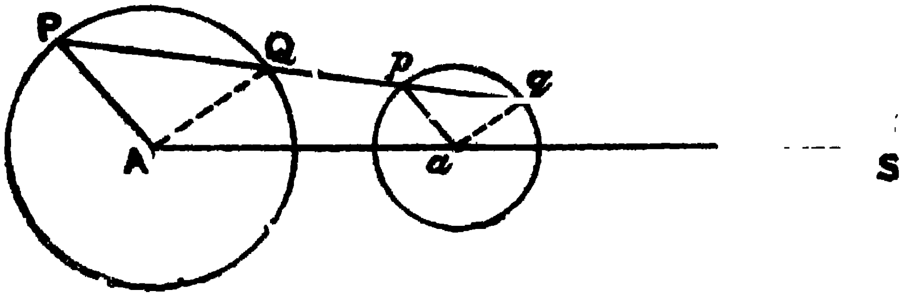
Ex. 23 (1). The join of the points of contact of any two parallel tangents to two circles passes through a centre of similitude.

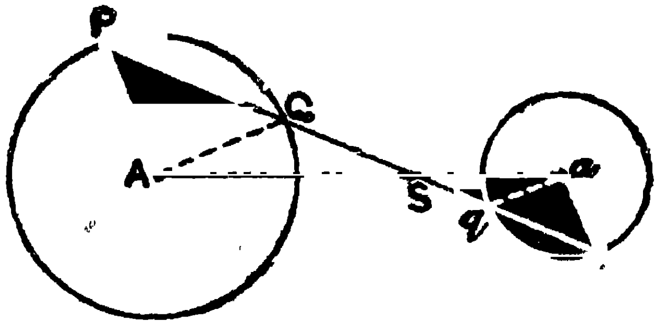
For the radii to the pts. of contact being perp. to the tangents are parallel.

Def. If through a centre of similitude of two circles, any straight line is drawn to cut the circles, then of the four points of section, the pair of points (one on each circle), that are nearer to the centre of similitude are called **corresponding points**, as **Q**, *q* ; also the pair of remoter points (one on each circle) are called **corresponding points**, as **P**, *p*. [Figs. of (28).]

A nearer point on one circle and a remoter point on the other are called *non-corresponding points* ; as **P**, *q* or **Q**, *p*.

28. THEOREM. *If through a centre of similitude of two circles any straight line is drawn to cut the circles, the radii to a pair of corresponding points are parallel.*





Let S be a centre of similitude and any st. line SP cut the circles (A), (*a*) in the corresponding pts. P, *p*, and Q, *q*; and let AP, *ap* be the radii drawn to P, *p*.

To prove that AP is parallel to *ap*.

Proof. Join AQ, *aq*.

Because the ∠s APQ, AQP are equal and their sum is less than two rt. angles; ∴ the ∠ APQ is an acute angle.

Similarly the ∠ *apq* is an acute angle.

Then in the Δs SAP, S*ap*,

(i) the ∠ ASP = the ∠ *a*S*p*,

(ii) SA : S*a* = AP : *ap*,

and (iii) the ∠s APS, *ap*S are both acute; [*Proved.*

∴ the ∠ APS = the ∠ *ap*S. [V. 6, *Cor.*

Hence AP is parl. to *ap*.

Similarly it may be shewn that AQ is parl. to *aq*. Q. E. D.

Cor. 1. *If through a centre of similitude of two circles any straight line is drawn to cut the circles, the tangents at the corresponding points are parallel.*

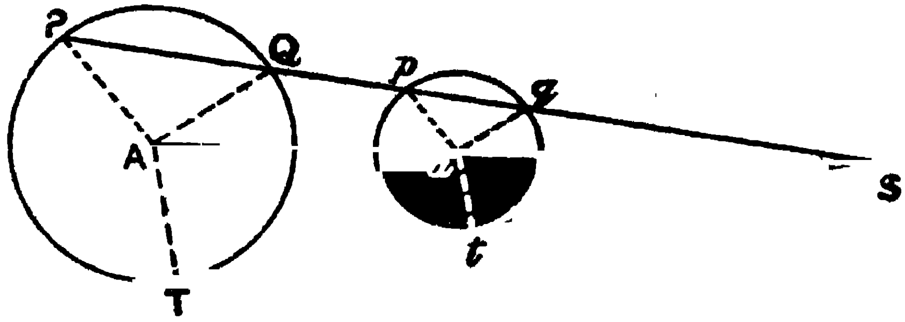
For the tangents are perp. to the radii drawn to the points.

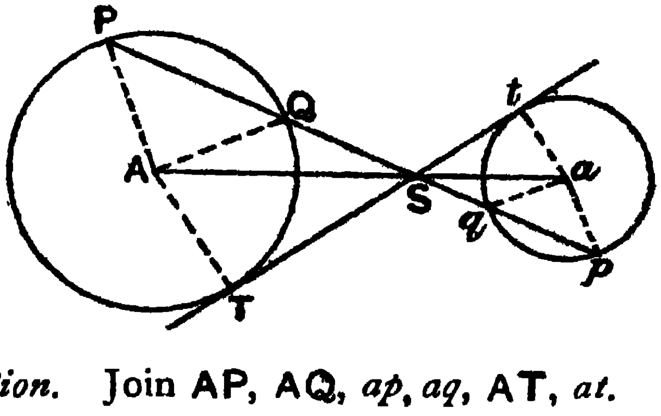
Cor. 2. *Any straight line, passing through a centre of similitude of two circles, is cut by the circles at the corresponding points in a* **fixed ratio,** *viz.*, the ratio of their radii.

For **SA** : **S***a* = **SP** : **S***p* = **SQ** : **S***q* = radius of (**A**) : radius of (*a*).

29. THEOREM. *If through a centre of similitude of two circles, any straight line is drawn to cut the circles at* P, Q *and* p, q *and another straight line to touch them at* T, t ; *prove that*

$$SP.Sq = Sp.SQ = ST.St = constant.$$





Construction. Join AP, AQ, *ap*, *aq*, AT, *at*.

Proof. Since S is a centre of similitude, the radii to corresponding points are parallel.

$$\therefore\ SP : Sp = SA : Sa = SQ : Sq\ ;$$

$$\text{and } ST : St = SA : Sa = SP : Sp.$$

Hence $SP.Sq = Sp.SQ$ ....(1);

and $ST.Sp = St.\ SP$ ....(2).

Also $ST^2 = SP.SQ$ ....(3).

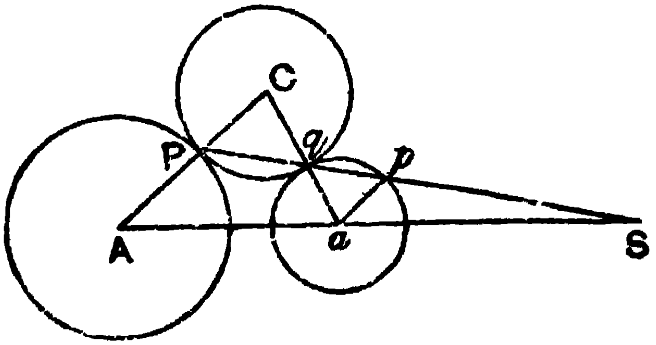
Dividing (2) by (3), $\dfrac{Sp}{ST} = \dfrac{St}{SQ}$;

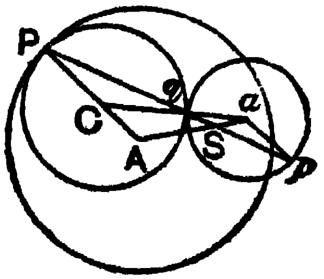
∴ $Sp.\ SQ = ST.St$; and $ST.St$ is constant.

Hence from (1), $SP.Sq = Sp.SQ = ST.St =$ constant.

Q. E. D.

30. THEOREM. *If a variable circle touches two fixed circles, the straight line joining the points of contact passes through a centre of similitude.*





Let the variable circle (C) touch the circles (A) and (*a*) at P and *q* respectively.

To prove that P*q* passes through a centre of similitude.

Proof. Let P*q* cut A*a* in S, and (*a*) again in *p*.

Join AP, *ap*, *aq*; then AP, *aq* pass through C.

Now the ∠CP*q* = the ∠C*q*P = the ∠*aqp* = the ∠*apq*;

∴ AP is parl. to *ap*.

Hence the △s SAP, S*ap* are similar;

∴ SA : S*a* = AP : *ap*;

∴ S is a centre of similitude. Q. E. D.

Obs. The centre of similitude is **external** or **internal** according as the contacts are of the **same** kind as in Fig. 1, or of **opposite** kinds as in Fig. 2.

Ex. 24. Draw diagrams to shew the centres of similitude of two circles (i) when they touch externally, (ii) when they touch internally, (iii) when they intersect, and (iv) when one is wholly within the other.

Ex. 25. Each vertex of a triangle is the internal centre of similitude of a pair of ex-circles [*App.* V. (27), *Obs.*]

Ex. 26. The internal and external centres of similitude of the in-circle and an ex-circle of a triangle are the point where the line of centres cuts the side that the ex-circle touches and the vertex opposite to that side.

Ex. 27. The centroid and the orthocentre of a triangle are the internal and external centres of similitude of the circum-circle and the nine-point circle.

VII. THE TANGENCIES

Many problems on the construction of circles have been given before [see p. 342]. We shall here supplement them by an interesting group of problems, known as the Tangency Problems or "The Tangencies."

In these Problems any **three** of the following **nine**—three points, three straight lines and three circles—being given, a circle is to be drawn to *pass* through the points and to *touch* the lines and the circles.

The following notation will help in remembering them. Let **P**, **L** and **C** stand for the words 'point,' 'straight line' and 'circle.' Then by adding the suffix 2 or 3 to **P**, **L** or **C**, we shall denote so many points, straight lines or circles.

Thus P_3 will symbolise the Problem "To draw a circle to pass through *three* given *points.*" ; **PLC** the Problem "To draw a circle to pass through a given *point*, and to touch a given *straight line* and a given *circle*" ; and so on.

There are altogether **ten** of these Problems.

31. PROBLEM 1. *To draw a circle to pass through three given points.* [P_3].

[This problem has been solved (see p. 344)].

Note. The problem has **one** solution. Why?

32. PROBLEM 2. *To draw a circle to touch three given straight lines.* [L_3].

[Already solved. See III. Problems 5 and 6.]

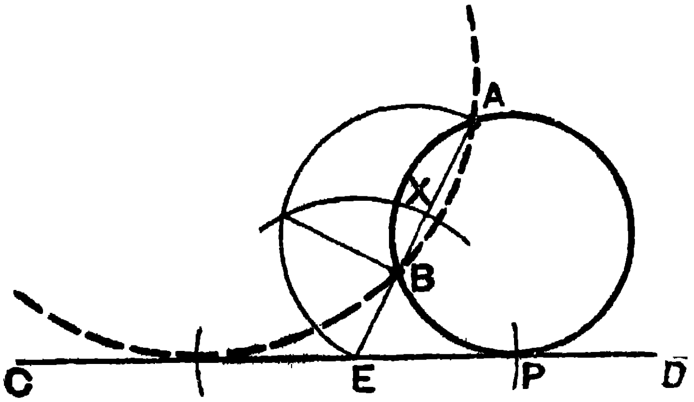
Note. The problem has **four** solutions. Why?

33. PROBLEM 3. *To draw a circle to pass through two given points and to touch a given straight line.* [P_2L.]

Let A, B be the given points, and CD the given straight line.

To draw a circle which shall pass through A and B, and touch CD.

FIRST METHOD



[**Analysis**. Suppose ABP to be the required circle. If AB be joined and produced to meet CD in E, then, because ED is a tangent, $EP^2 = EA.EB$. Hence EP is the mean proportional to EA and EB.

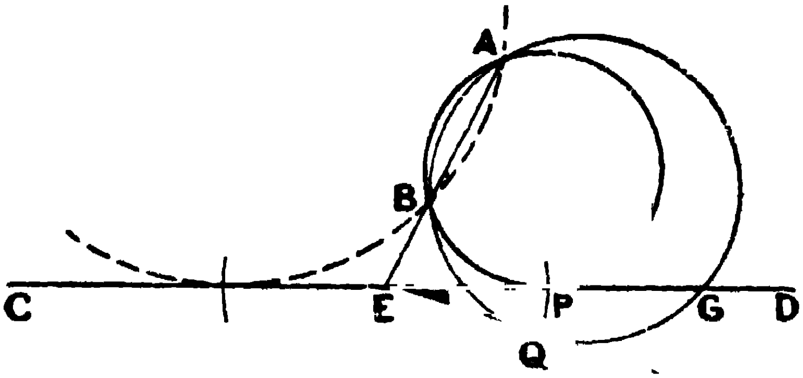
Thus we have the following]

Construction. Join AB and produce it to meet CD in E. Take EX, the mean proportional to EA and EB. From ED, cut off EP=EX. Draw a circle through A, B and P.

Then ABP is the required circle.

[Prove by IV. 13.]

SECOND METHOD



Construction. Join AB and produce it to meet CD in E. Draw a circle through A and B to cut CD in F and G. From E, draw a tangent EQ to this circle. From ED, cut off EP=EQ. Draw a circle through A, B and P.

Then ABP is the required circle.

[Prove by IV. 13.]

Note. Since P may be taken on either side of E, there is another solution. Give a construction for this case.

Thus the Problem has **two** solutions.

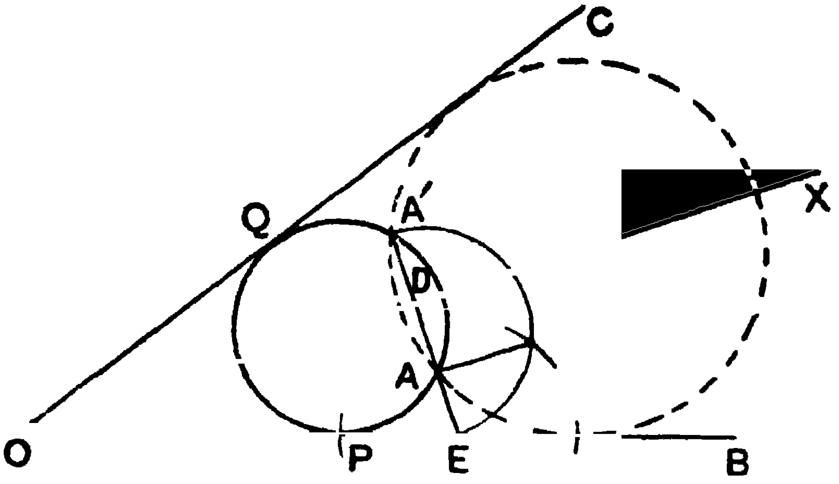
Question. Illustrate by figures the cases—(i) when one of the points A is in CD and (ii) when AB is parallel to CD.

[(i) Draw AO perp. to CD, and let the perp. bisector of AB cut AO in O. (ii) Draw the perp. bisector of AB to meet CD in P.]

34. PROBLEM 4. *To draw a circle to pass through a given point and to touch two given straight lines.* [PL_2.]

Let A be the given point, and OB, OC the given straight lines.

To draw a circle which shall pass through A and touch OB and OC.



[**Analysis**. Suppose APQ to be the required circle. Since the locus of the centres of circles which touch OB, OC is the bisector of the ∠O [p. 343, (7)], the centre of ⊙APQ is in the bisector OX; and thus the ⊙ is symmetrical about OX. Hence ∵ the ⊙ passes through A, it also passes through A′, the image of A in OX. Let A′A produced meet OB at E; then $EP^2 = EA.EA'$. Thus EP is a mean proportional to EA and EA′.

Hence the]

Construction. Draw OX bisecting the ∠O. From A, draw AD perp. to OX, and make DA′ = DA. Produce A′A to cut OB in E. Take EP a mean proportional to EA and EA′; so that $EP^2 = EA.EA'$. Draw a circle through A, A′ and P.

Then AA′P is the circle required.

Proof. Now $EA.EA' = EP^2$;

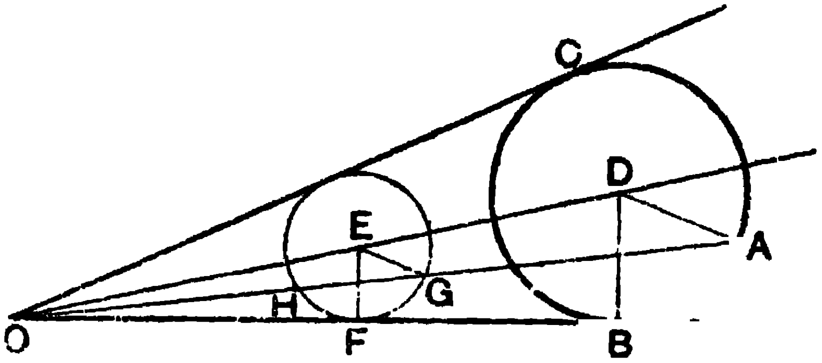
∴ ⊙AA′P touches OB at P; [IV. 13.

and since it is symmetrical about OX, it also touches OC.

Note. Because P may be on either side of E, the problem has another solution. Give a construction for this solution.

Thus the problem has **two** solutions.

[ALTERNATIVE METHOD]



Construction. Draw OD bisecting the $\angle$ O. Then OD is the locus of the centres of all circles that touch OB and OC. Take any point E in OD. Draw EF perp. to OB. Then the circle whose centre is E and radius EF touches OB and OC.

Join OA, cutting the circle (E) in G and H.

Join EG; and through A, draw AD parl. to GE, meeting OD in D.

With centre D and radius DA, draw the ⊙ ABC.

Then ABC is the required circle.

Proof. Draw DB perp. to OB.

Now the $\triangle^s$ OEG and ODA are simlr.; so also are the $\triangle^s$ OEF and ODB;

$$\therefore \mathrm{EG} : \mathrm{DA} = \mathrm{OE} : \mathrm{OD} = \mathrm{EF} : \mathrm{DB};$$

$$\therefore \mathrm{EG} : \mathrm{DA} = \mathrm{EF} : \mathrm{DB}; \text{ or } \mathrm{EG} : \mathrm{EF} = \mathrm{DA} : \mathrm{DB};$$

$$\therefore \mathrm{DA} = \mathrm{DB}.$$

Hence the circle, with centre D and radius DB or DA, touches OB and OC.

[The second solution is obtained by drawing through **A**, a parallel to **HE**. Give a construction for this solution.]

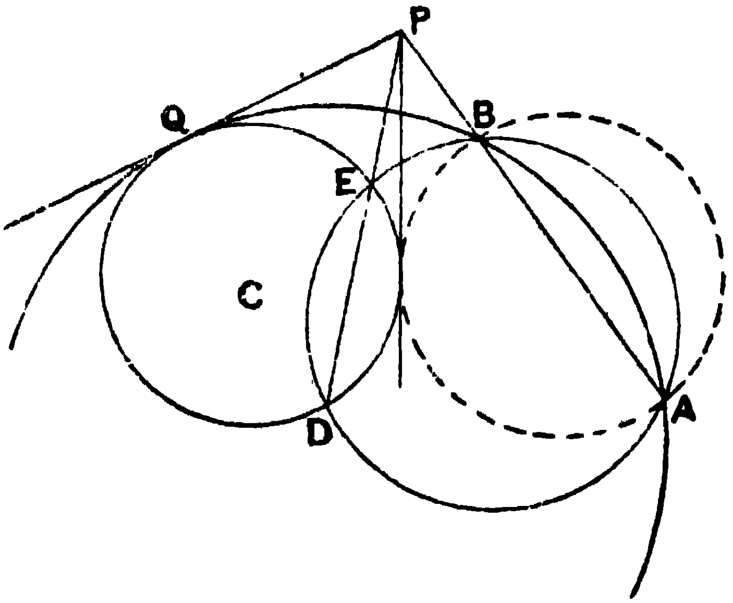
Question. Give a construction for the cases—(i) when the given point **A** is in one of the given lines **OB**, and (ii) when the given lines are parallel.

[(i) The centre of required circle is the point where the perp. to **OB** at **A** cuts **OD**. (ii) Find **XY**, locus of the centre by (8), p. 343. Required circle is that which passes through **A** and its image in **XY** and touches one of the given lines.]

35. PROBLEM 5. *To draw a circle to pass through two given points and to touch a given circle* [P_2C.]

Let A, B be the given points and (C) the given circle.

To draw a circle which shall pass through A and B and touch (C).



[**Analysis.** Let the ⊙ ABQ through A and B touch (C) internally at Q.

Let the common tangent at Q meet AB produced at P. Draw a secant PED to cut (C) in D and E.

Then ∵ $PA.PB = PQ^2 = PD.PE$, ∴ A, B, E, D lie on a circle.

Hence we have the following]

Construction. Through A and B, draw a circle to cut (C) at D and E. Join AB, DE, and produce them to meet at P. From P, draw a tangent PQ to the circle (C). Draw a circle through A, B and Q.

Then this circle shall touch (C) at Q.

[Prove by IV. 13.]

Note. Since another tangent can be drawn from P to (C), a second circle can be drawn to touch (C) externally.

Thus the prob'em has **two** solutions.

36. PROBLEM 6. *To draw a circle to touch two given straight lines and a given circle.* [L,C.]

Let A and B be the given straight lines and (C) the given circle.

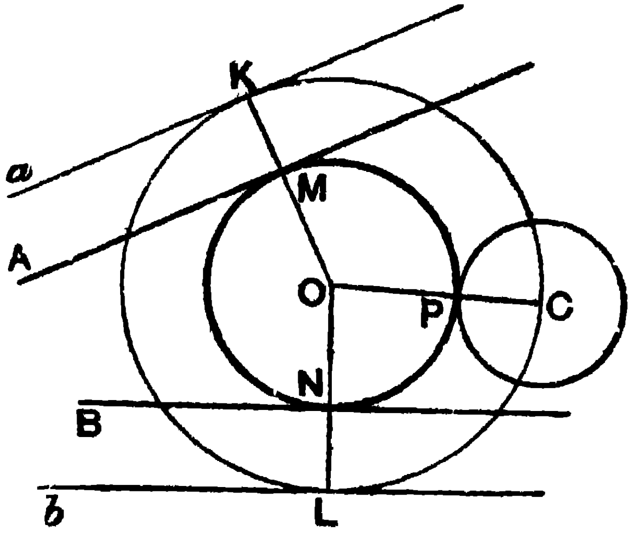
To draw a circle to touch A, B and (C).

[**Analysis**. Suppose the ⊙ MNP of centre O to touch A and B at M and N, and (C) at P.

Join OC ; then OC passes through P.

Produce OM, ON to K and L.
and make MK = PC = NL ; so that OC = OK = OL.

Draw *a*K parl. to AM and *b*L to BN.



Then the ⊙ with centre O and radius OC passes through C, K, L and touches *a*K and *b*L at K and L.

Hence the following]

Construction. Draw *a* parl. to A, and *b* to B and on the sides of A and B remote from C, at a distance equal to the radius of (C).

Draw a ⊙ to pass through C and to touch *a* at K and *b* at L [*Tang. Prob.* 4]. Let O be the centre of this circle.

Join OK, OL, OC cutting A at M, B at N, and (C) at P.

With centre O and radius OP, draw a circle PMN.

Then PMN is the circle required.

Proof. Because OC = OK = OL, and CP = MK = NL;

∴ OP = OM = ON.

∴ the circle passes through M and N;

and it touches A and B at M and N, for the angles at M and N are right angles.

And because (O) and (C) meet at P on the line of centres;

∴ (O) touches (C) at P. [III. 16A.

Note. Since two circles can be drawn to pass through C and to touch *a* and *b*, there are *two* solutions, both circles touching (C) *externally*. Also the parallels *a* and *b* can be drawn on the same side as C; hence there are *two* more solutions, the circles touching (C) *internally*.

Thus the problem has **four** solutions.

37. PROBLEM 7. *To draw a circle to pass through a given point and to touch a given straight line and a given circle.* [PLC].

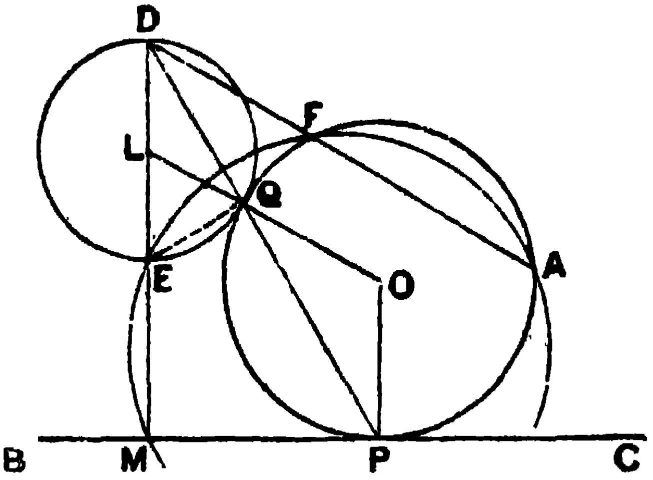
Let A be the given point, BC the given straight line and (L) the given circle.

To draw a circle which shall pass through A and touch BC and (L).

[**Analysis.** Suppose APQ to be the circle that passes through A and touches BC at P and (L) at Q.

Let O be the centre of this ⊙. Join OP; then OP is perp. to BC.

Join PQ, and produce it to cut (L) at D. Draw the diameter DE, and produce it to meet BC at M. Then DM is parl. OP [(LVI), 10]; thus DM *is perpendicular to* BC.



Join AD cutting the ⊙ APQ at F.

Joint EQ ; then the ∠ˢ DQE, EMP being rt. angles, PQEM is a cyclic quadl. Thus DE.DM = DQ.DP = DF.DA ; hence, A, F, E, M *are concyclic.*

Hence the following]

Construction. Draw the diameter DE perp. to BC, meeting BC at M. Join AD ; and through A, E and M, draw a circle, cutting AD at F. Through A and F, draw a circle to touch BC at P [*Tang. Prob.* 3] ; and let O be the centre of this circle.

Then (O) is the required circle.

Proof. Join EQ, DP ; and let DP cut the ⊙ (L) in Q.

Now because the ∠ˢ DQE, EMP, are right angles ;

∴ PQ EM is a cyclic quadrilateral.

Hence DQ.DP = DE.DM = DF.DA ;

∴ Q is a point on the circle (O).

Join OP, OQ, LQ.

Then OP is perp. to BC and parl. to DM ;

∴ the ∠ OPD = the alt. ∠ PDM.

Hence the ∠ OQP = the ∠ OPQ = the ∠ LDQ = the ∠ LQD

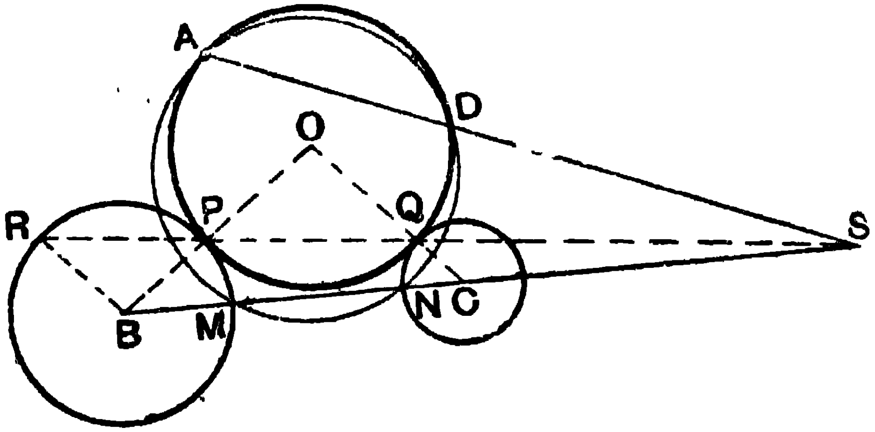
∴ OQL is a st. line. [(II), 1.

Thus the circles (L) and (O) meet Q on the line of centres ; hence they touch one another at Q. [III. 16A.

38. Problem 8. *To draw a circle to pass through a given point and to touch two given circles.* [PC_2].

Let A be the given point, and (B) and (C) the given circles.

To draw a circle which shall pass through A and touch (B) and (C).



[**Analysis.** Suppose the ⊙ APQ to pass through A and touch (B) and (C) at P and Q.

Because the ⊙ APQ touches (B) and (C) at P and Q, BC and PQ *intersect at a centre of similitude* ; call it S.

[*App.* V. (30)].

Let BS cut (B) at M and (C) at N. Join AS, cutting the ⊙ APQ at D. Then SM.SN = SP.SQ [*App.* V. (29)] = SA.SD. Thus A, D, N, M *are concyclic.*

Hence the following]

Construction. Join BC, and divide it externally at S in the ratio of the radii of (B) and (C) ; thus S is the external centre of similitude.

Let BS cut the circles at M and N. Join AS, and through A, M and N, draw a circle cutting AS at D.

Draw a circle through A and D to touch (B) in P [*Tang. Prob.* 5]. Let O be its centre.

Then (O) is the circle required.

Proof. Join SP, cutting (B) again at R and (C) at Q.

Join RB, OB, OQ and CQ; then OB passes through P.

Since S is a centre of similitude of (B) and (C).

∴ SP.SQ = SM.SN = SA.SD; [*App.* V. (29).

∴ Q is a point on the circle (O).

And ∵ SQR passes through a centre of similitude;

∴ BR is parl. to CQ; [*App.* V. (28).

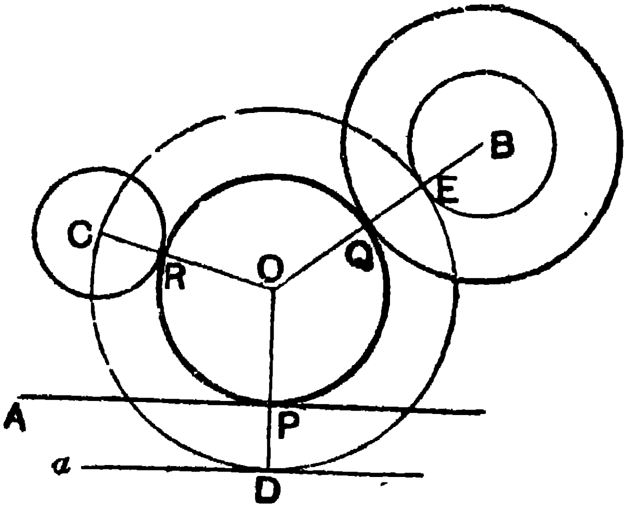
∴ ∠ SQC = ∠ SRB = ∠ BPR = ∠ OPQ = ∠ OQP.

∴ OQC is a straight line.

Hence the circles (O) and (C) meet at Q on the line of centres;

∴ they touch one another at Q. [III. 16A.

39. PROBLEM 9. *To draw a circle to touch a given straight line and two given circles.* [LC₂.]



Let A be the given st. line, and (B) and (C) the given circles, of which let (B) > (C).

To draw a circle which shall touch A and the circles (B) and (C).

Construction. On the side of A remote from B, draw *a* parl. to A at a distance equal to the radius of (C).

With centre B and radius equal to the difference of the radii of (B) and (C) describe a circle.

Draw a circle to pass through C and to touch *a* at D and the last drawn ⊙ at E [*Tang. Prob.* 7]. Let O be the centre of this circle.

Join OD cutting A at P, OB cutting (B) at Q and OC cutting (C) at R. Thus E is on O3.

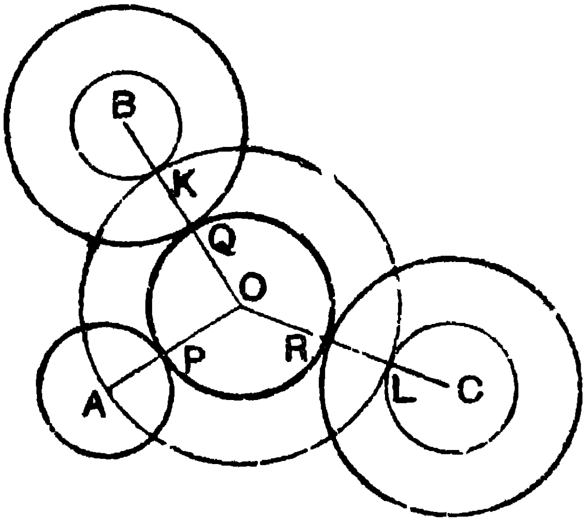
With centre O and radius OP, draw the circle PQR.

Then PQR is the required circle.

[Supply Proof.]

40. PROBLEM 10. *To draw a circle to touch three given circles.* [C$_8$].

Let (A), (B) and (C) be the given circles ; and let (A) be not greater than either (B) or (C).



To draw a circle which shall touch (A), (B) and (C).

Construction. With centre B, and radius equal to the difference of the radii of (A) and (B), draw a circle.

With centre **C** and radius equal to the difference to the radii of (**A**) and (**C**), draw a circle.

Through **A**, draw a circle to touch the two last drawn circles at **K** and **L** [*Tang. Prob.* 8]. Let **O** be the centre of this circle.

Join **OA** cutting (**A**) in **P**, **OB** cutting (**B**) in **Q** and **OC** cutting (**C**) in **R**; thus **K** and **L** are on **OB** and **OC**.

With centre **O** and radius **OP**, draw the circle **PQR**.

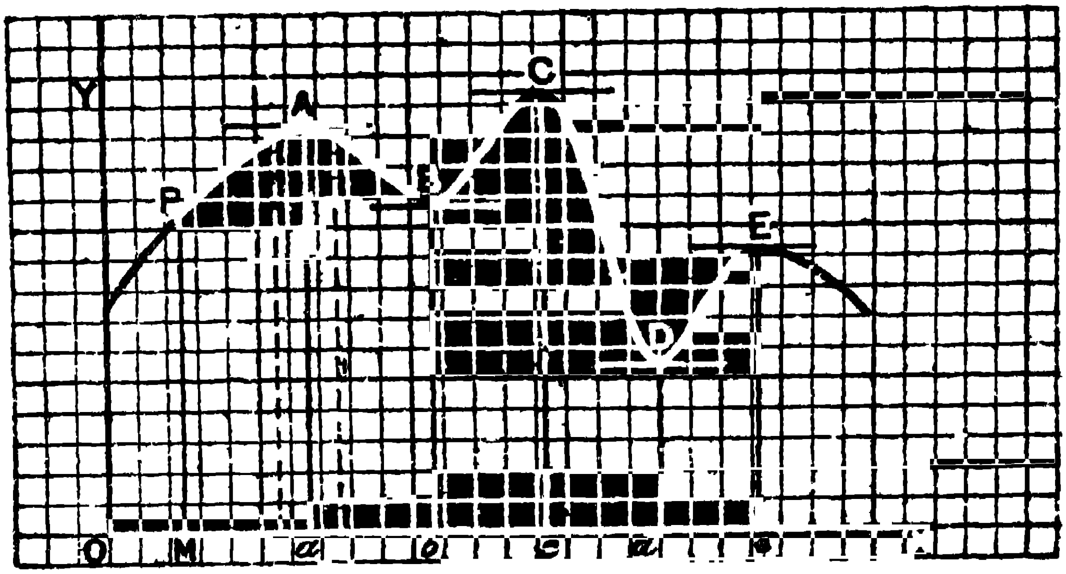
Then **PQR** is the required circle.

[Supply Proof.]

VIII. MAXIMA AND MINIMA

A magnitude is said to *vary continuously* when as it passes from any one value to any other, it receives all intermediate values throughout the change.

Let **OX** and **OY** be two fixed straight lines at right angles, and **PE** be the graph of a moving point **P**.



Let the ordinate **PM** represent the value of a variable geometrical magnitude.

As P approaches A, PM continually increases till P reaches A and then it begins to decrease. Hence at A, PM is said to have a maximum value A*a*.

Again as P approaches B, PM gradually decreases till P is at B, and then it begins to increase. Thus at B, PM is said to have a minimum value B*b*.

Similarly PM has maxima values at C and E and a minimum value at D.

It is thus seen that maxima and minima values of a variable magnitude occur *alternately*, hence between two successive *maxima* there is a *minimum* ; and between two successive *minima* there is a *maximum*.

It is also seen that each of the points of maximum or minimum value is a *turning point* of the curve at which the tangent is parallel to OX.

From the figure it is also evident that $Bb > Ee$, that is, a minimum value of PM is greater than a maximum value. Thus a maximum value does not necessarily mean that it is the *greatest possible value* of PM, but it simply means that A*a* *is the greatest of all values of* PM *in the neighbourhood of* A *and on either side of* A. Similar remarks apply to a minimum value.

Again it is clear that as P approaches A, the value of PM increases and when P passes through A it decreases. Hence in the immediate vicinity of A, PM has two positions on either side of A in which PM has two equal values. Similarly when PM attains a minimum value B*b*, it has two positions near B and on either side of B in which it has two equal values.

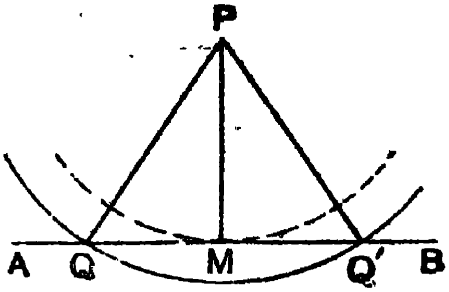
Thus a maximum or a minimum value of a variable magnitude lies between any pair of equal values, and may be found by making these two values approach infinitely close to one another.

This principle is of great use in discovering a maximum or a minimum value of a variable.

For example—*To find the least possible straight line that can be drawn to a given straight line* AB *from a given point* P *outside it.*

Draw a circle with centre **P** to cut **AB** in **Q**, **Q′**; thus **PQ = PQ′**.

Hence at some point **M** between **Q** and **Q′**, the straight line shall have either a maximum or a minimum value.



Now as the radius of the circle diminishes, **Q** and **Q′** approach one another, and ultimately coincide at **M** where the circle touches **AB**. Draw **PM**.

Thus **PM** is a maximum or a minimum.

Having discovered that **PM** is either a maximum or a minimum, we find from the figure which of the two values **PM** has.

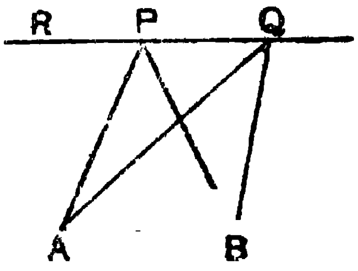
Thus **PM** is a radius to the point of contact of the circle which touches **AB**; hence **PM** is perp. to **AB**; and we see that every other line from **P** to **AB** cuts the circle and is therefore greater than **PM**. Hence **PM** is the least of all.

41. PROBLEM. *To find a point in a given straight line such that the sum of its distances from two given points is a minimum.* [*App.* I. (10)]

Observe that **AP + BP is a minimum when AP and BP make equal angles with XY.**

42. THEOREM. *Of all triangles on the same base and of equal area, the isosceles triangle has the least perimeter.*
[*App.* I. Ex. 7.]

Let **ABP** be an isosceles triangle on the base **AB**, and let **ABQ** be any triangle on **AB** equal in area to the △**ABP**.



To prove that **AP + BP** is a minimum.

Const. Join **PQ**; and produce it to **R**.

Proof. Because $\triangle ABP = \triangle ABQ$; $\therefore$ RPQ is $\parallel$ to AB.

$$\therefore \angle RPA = \angle PAB = \angle ABP = \angle BPQ.$$

Thus AP, BP make equal angles with RPQ;

$\therefore$ AP + BP is a minimum. Q. E. D.

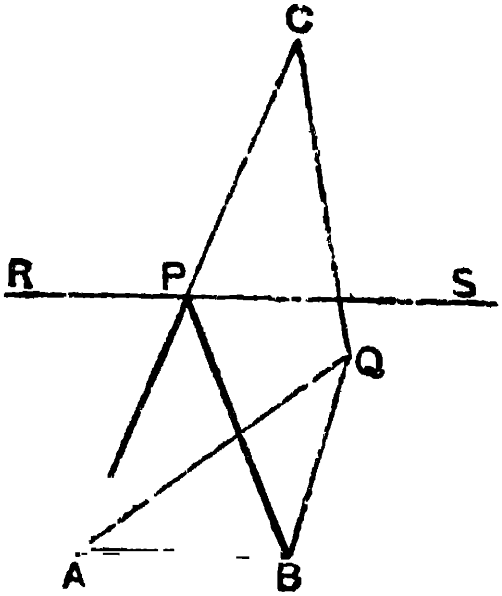
Ex. 28. Of all triangles of given base and area, the equilateral triangle has the minimum perimeter.

43. THEOREM. *Of all triangles on the same base and of given perimeter, the isosceles triangle has the maximum area.*

Let ABP be an isosceles triangle on the base AB; and let ABQ be any triangle of equal perimeter and on the same base.

To prove that

$$\triangle ABP > \triangle ABQ.$$



Const. Produce AP to C and make PC = PB. Join CQ. Draw RPS parl. to AB.

Proof. Now $AQ + BQ$
$= AP + BP = AP + PC$
$= AC < AQ + CQ$;

that is, $AQ + BQ < AQ + CQ$;

$$\therefore BQ < CQ.$$

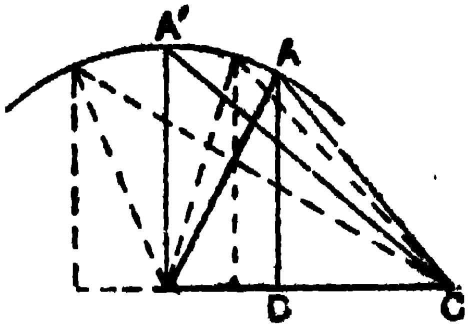
Hence Q is below the parallel RPS;

$\therefore$ the alt. of $\triangle ABP >$ the alt. of $\triangle ABQ$;

$\therefore \triangle ABP > \triangle ABQ.$ Q. E. D.

Note. From the figure it is seen that Q cannot fall on or above the parallel RPS, for then in each case AQ + BQ would be greater than AP + BP.

44. THEOREM. *Two sides of a triangle being given, to prove that its area is a maximum when the sides include a right angle.* [MRS. EX. BK. II, 25].



Let the sides AB, BC of the Δ ABC be given.

To prove that the area of the Δ ABC is greatest when AB, BC are at right angles.

Const. Draw AD perp. to BC.

Proof. Then the Δ ABC $=\frac{1}{2}$ BC.AD.

Thus BC being constant, the area of the Δ ABC is maximum, when AD is maximum.

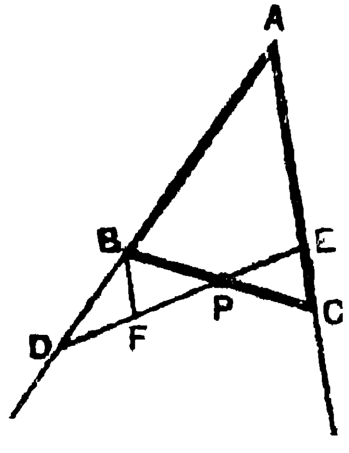
Now AD increases as the ∠ ABC increases, till it is a rt. angle; and when the ∠ ABC is a rt. angle, that is, when AB attains the position A'B, then AD coincides with A'B and has its greatest value A'B.

Hence when AB, BC include a rt. angle, the area of the Δ ABC is greatest. Q. E. D.

Ex. 29. Of all parallelograms which can be formed with given adjacent sides, the rectangle has the greatest area. [(XLIX), 37.]

Ex. 30. Of all parallelograms of given diagonals, the rhombus has the maximum area. [Mis. Ex Bk. II, 26.]

45 PROBLEM. *Through a given point within a given angle to draw a straight line so as to form with the arms of the angle a triangle of minimum area.*



Let BAC be the given angle, and P the given point within it.

To draw a straight line BC *through* P, so that Δ ABC may be a minimum.

Const. Through P, draw a st. line DPE, meeting AB in D and AC in E. Make PF = PE; and through F, draw FB parl. to AC to cut AB in B. Join BP, and produce it to meet AC in C.

Then BC is the st. line required.

Proof. In the $\triangle^s$ BPF, CPE,

PF = PE, $\angle$BPF = $\angle$CPE, and $\angle$BFP = $\angle$CEP;

$\therefore$ BP = CP; and the $\triangle$ BPF = the $\triangle$ CPE.

To each of these equal triangles, add the fig. ABPE;

$\therefore$ the $\triangle$ ABC = the quadl. ABFE;

$\therefore$ the $\triangle$ ABC $<$ the $\triangle$ ADE.

Note. Of all the st. lines through **P**, the required line is that which **is bisected at P.**

46. THEOREM. *If a straight line is divided into any two parts, then* (i) *the rectangle contained by the two parts is a maximum and* (ii) *the sum of their squares is a minimum, when the two parts are equal*

One solution is already given [see *App.* IV. (4)].

Otherwise thus :—Let a and b be the two parts; so that $a+b$ is constant, whatever be the magnitudes of a and b

(i) We have $4ab=(a+b)^2-(a-b)^2$ identically;

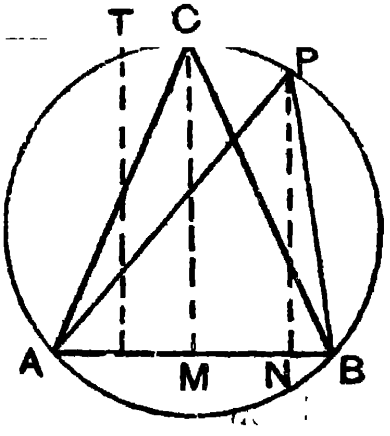
$$\therefore\ ab=\tfrac{1}{4}(a+b)^2-\tfrac{1}{4}(a-b)^2.$$

Now $a+b$ [and therefore $\frac{1}{4}(a+b)^2$] is constant; hence ab is maximum, when $\frac{1}{4}(a-b)^2=0$; that is, when $a=b$.

(ii) Again $a^2+b^2=\frac{1}{2}(a+b)^2+\frac{1}{2}(a-b)^2$ identically.

Now $a+b$ [and therefore $\frac{1}{2}(a+b)^2$] is constant; hence a^2+b^2 is minimum, when $\frac{1}{2}(a-b)^2=0$; that is, when $a=b$.

47. THEOREM. *Of all triangles which can be inscribed in a circle on a given chord as base, the isosceles triangle has the maximum area.*



Let ABC be an isosceles $\triangle$ inscribed in a circle; and APB be any other $\triangle$ inscribed in the same circle.

To prove that $\triangle$ ABC $>$ $\triangle$ ABP.

Const. At C, draw the tangent TC; and let CM, PN be perp. to AB.

Proof. Because AC = BC ; ∴ C is the middle pt. of the arc ACB ; thus TC is ∥ to AB.

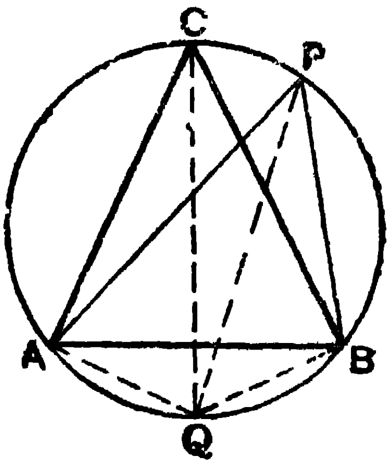
Now TC being a tangent, every pt. T in it except C is outside the circle ; ∴ the perp. from T to AB > the perp. from any pt. in the arc ACB, except C.

But the perp. from T to AB = CM ; ∴ CM > PN.

And △ ABC = $\frac{1}{2}$AB.CM, and △ ABP = $\frac{1}{2}$ AB.PN ;

∴ △ ABC > △ ABP, for CM > PN. Q. E. D.

48. THEOREM. *Of all triangles which can be inscribed in a circle on a given chord as base, the isosceles triangle has the maximum perimeter.*



Let ABC be an isosceles △ inscribed in a circle on the chord AB as base ; and let APB be any other △ inscribed in the circle on the same base.

To prove that

AC + BC > AP + BP.

Const. Draw CQ perp. to AB cutting the circle at Q ; so that CQ is a diameter.

Join PQ, AQ, QB.

Proof. By Ptolemy's Theorem,

QB.AC + AQ.BC = AB.CQ, or AQ(AC + BC) = AB.CQ ;
and QB.AP + AQ.BP = AB.PQ, or AQ(AP + BP) = AB.PQ.

∴ AC + BC : AP + BP = CQ : PQ

But CQ > PQ ; ∴ AC + BC > AP + BP. Q. E. D.

Ex. 31. The rectangle of maximum area and perimeter which can be inscribed in a circle is a square.

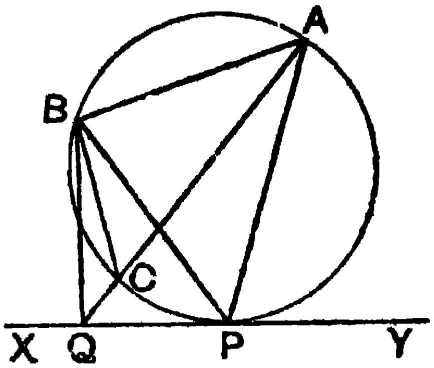
Ex. 32. Of all triangles inscribed in a circle, the equilateral triangle is of maximum area and perimeter.

49. PROBLEM. *A and B are fixed points on the same side of a given line XY; to find a point in XY at which AB subtends a maximum angle.*

Construction. Draw a circle to pass through A and B, and to touch XY at P [*App.* V. (33)].

Then P is the required point.

Proof. Let Q be any other point in XY. Join AP, BP, AQ, BQ.



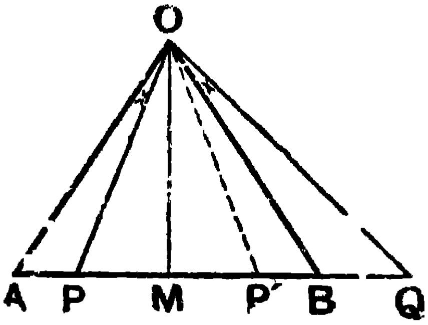
The construction is proved by shewing that the ∠APB > the ∠AQB.

Join BC. Then ∠APB = ∠ACB in the same segment. But ∠ACB > ∠AQB; ∴ ∠APB > ∠AQB.

50. THEOREM. *Of all triangles having a given vertical angle and altitude, the isosceles triangle has the least area.*

Let AOB be an isosceles triangle, and POQ any other triangle, both having the same altitude OM and equal vertical angles, so that ∠AOB = ∠POQ.

To prove that △ AOB < △ POQ.



Const. Make MP′ = MP and join OP′.

Proof Then △s OMP, OMP′, are congruent, so that ∠POM = ∠P′OM; hence ∠AOP = ∠BOP′ and AP = BP′.

Also ∠AOB = ∠POQ; ∴ ∠AOP = ∠QOB;

hence ∠BOP′ = ∠QOB.

∴ OP′ : OQ = BP′ : BQ = AP : BQ.

But OP′ < OQ [I. 12A]; ∴ AP < BQ.

Thus AB < PQ;

∴ △AOB < △POQ. Q. E. D.

Ex. 33. **AB** is a diameter of a circle and **BA** produced passes through a fixed point **P** outside the circle ; prove that **PA** is the least and **PB** the greatest of all straight lines which can be drawn from **P** to the circle.

Ex. 34. **AB** is a diameter of a circle of centre **C** and **P** a fixed point in **AB** between **B** and **C** ; prove that **PA** is the greatest and **PB** the least of all straight lines which can be drawn from **P** to the circle.

Ex. 35. Find the minimum distance between two fixed circles that do not intersect.

Ex. 36. Given the rectangle contained by two straight lines, the sum of the squares on the lines is minimum, when they are equal.

Ex. 37. Of all rectangle of given perimeter, the square has the maximum area.

Ex. 38. Of all rectangle of given area, the square has the minimum perimeter.

Ex. 39. If one circle is within another, find the maximum and minimum chords of the outer circle that touch the inner.

Ex. 40. Of all quadrilaterals of given diagonals, the one whose diagonals are at right angles, is maximum.

[*Hint.* Use *App.* II. (2), and *App.* V. (44).]

Ex. 41. Of all triangles which can be inscribed in a given triangle, the pedal triangle has the least perimeter.

[*Hint.* The sides of a pedal Δ are equally inclined to the sides of original Δ (*App.* III Ex. 10)].

IX. POLE AND POLAR

Def. If on any straight line drawn from the centre of a circle, two points are taken such that *the rectangle contained by their distances from the centre is equal to the square on the radius*, then the perpendicular to the line at either point is called the **polar** of the other point with respect to the circle, and the point with regard to the polar is called its **pole**.

Thus if P, Q are points in OP [see Fig. of (51)] such that OP.OQ = (radius)^2, then

(i) if PX is perp. to OP at P, PX is the polar of Q and Q is the pole of PX ; and

(ii) if AB is perp. to OP at Q, AB is the polar of P and P is the pole of AB.

Note. 1. Of the two points **P** and **Q**, either is said to be the **inverse** of the other. Hence the polar of a point is the perpendicular at its *inverse.*

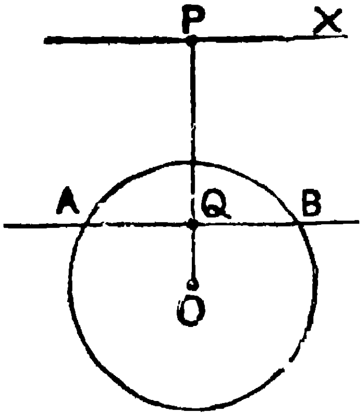
Note. 2. We have **OP.OQ**=(radius)^2. Thus the radius of the circle is intermediate in magnitude between **OP** and **OQ**. Hence if one of the inverse points **P** and **Q** be inside the circle, the other must be outside it.

Note. 3. Since **OP.OQ**=(radius)^2, therefore **OP.OQ** is *constant* Hence as **OQ** diminishes, **OP** increases in magnitude; and *vice versâ.* Thus when **Q** coincides with **O**, **OP** becomes infinite.

Note 4. Again **OP.OQ**=(radius)^2, and this is the case when **OP**=**OQ**=radius. Hence when the points **P** and **Q** coincide on the circle, the polar of either becomes the tangent at the point of coincidence.

Thus **the polar of a point on a circle is the tangent to the circle at the point.**

51. PROBLEM. *To construct the polar of a given point with respect to a given circle.*



Let (O) the given circle of radius r.

First, let the given point be outside the circle, as **P**.

Construction. Join **OP** and divide **OP** at **Q** so that the rect. **OP,OQ**$=r^2$. At **Q**, draw **AQB** perp. to **OP**.

Then **AB** is the polar of **P**.

Secondly, let the given point be inside the circle, as **Q**.

Construction. Join **OQ** and produce **OQ** to **P** so that **OP.OQ**$=r^2$. Draw **PX** perp. to **OP** at **P**.

Then **PX** is the polar of **Q**.

52. THEOREM. *The chord of contact of the tangents to a circle from an external point is the polar of the point with respect to the circle.*

Let AB be the chord of contact of the tangents PA, PB from the pt. P to the circle (O) [See Fig. of III. 15].

To prove that AB is the polar of P.

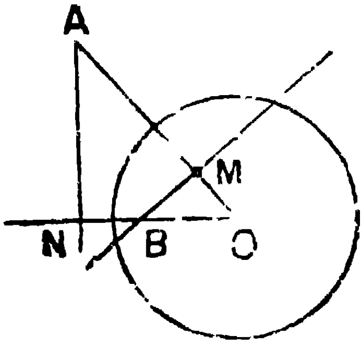
Let AB, OP intersect at M. Then AM is perpendicular to OP ; and it is drawn from the rt. angle A to the hypotenuse OP ; $\therefore$ $OA^2 = OM.OP$; hence AB is the polar of P.

Q. E. D.

Cor. *If two circles cut orthogonally, the pole of their common chord with respect to either is the centre of the other.*

For if (A) and (B) cut orthogonally at P and Q, then by *App.* V. (17), AP and AQ are tangents to (B).

53. THEOREM. *If the polar of a point* A *with respect to a circle passes through a point* B, *then the polar of* B *passes through* A.



Let O be the centre of the ⊙ and r its radius ; and let BM the polar to A pass through B.

To prove that the polar of B passes through A.

Construction. Join OB and draw AN perp. to OB.

Proof. Because BM is the polar of A ;

$\therefore$ the angles at M are rt. angles, and $OM.OA = r^2$.

Also the angles at M and N are right angles ;

$\therefore$ A, N, B, M are concyclic ;

$\therefore$ $OB.ON = OM.OA = r^2$;

∴ AN is the polar of B.

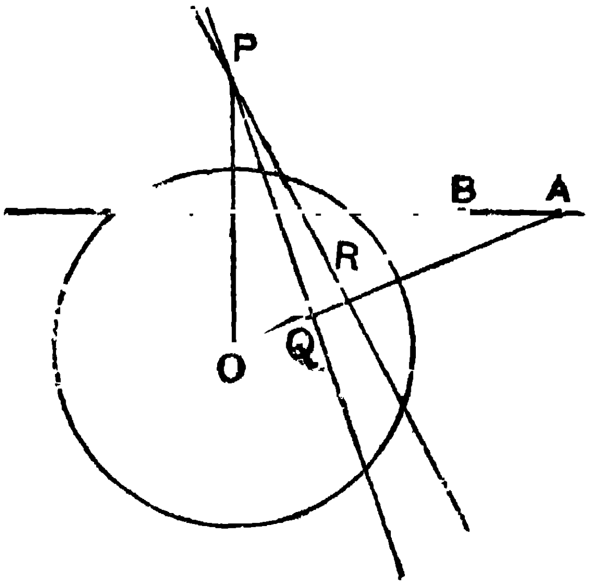
And it passes through A Q. E. D.

Ex. 42. Prove this Theorem when **A** and **B** are both outside the circle.

Ex. 43. Prove that the angle between the polars of **A** and **B** is equal, or supplementary, to the angle between **OA** and **OB**.

Def. If two points are such that the polar of either passes through the other, they are called **conjugate points** with respect to the circle, and their polars are called **conjugate lines.**

54. THEOREM. *The point of intersection of the polars of any two points is the pole of the line joining the points.*



Let QP, RP be the polars of A and B ; and let them intersect at P.

To prove that P is the pole of the line AB.

Proof. Since the polar of A passes through P ;

∴ the polar of P passes through A. [*App.* V. (53)

For the same reason, the polar of P passes through B.

Thus the polar of P passes through both the pts. A, and B ;

∴ AB is the polar of P ; that is, P is the pole of AB.

Q. E. D.

We may enunciate this Theorem otherwise thus :—*The point of intersection of any two lines is the pole of the line joining their poles.*

Note. App. V. (52) may follow as a Corollary to this Theorem. For **AP, BP,** the polars of **A** and **B,** intersect at **P** ; therefore **P** is the pole of the line joining **AB.**

Ex. 44. From an external point **A,** tangents **AM, AN** are drawn to a circle ; if the tangent at any point **B** on the circle cuts **MN** produced at **P,** shew that **P** is the pole of **AB.** [See *Note* 4, *Def.* of Pole and Polar.]

Ex. 45. The locus of the intersection of the tangents to a circle at the extremities of a chord which passes through a fixed point, is the polar of the point.

55. THEOREM. *If through a given point any straight line is drawn to cut a circle, the line is divided harmonically by the circle, the point and the polar of the point.*

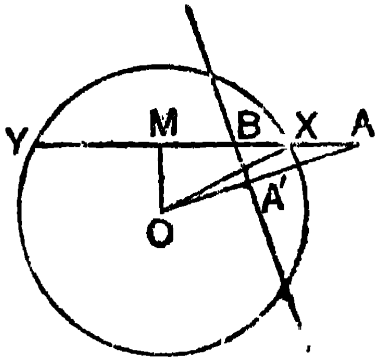


Fig. 1.

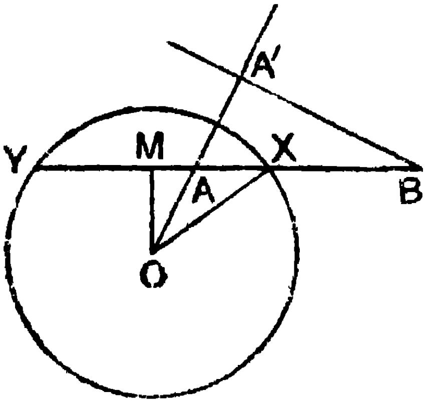


Fig. 2.

Let A be a pt. and AXBY a st. line through A, cutting the ⊙ of centre O, at X and Y and the polar of A, at B. Let A' be the inverse of A.

To prove that the line is cut harmonically at A, X, B and Y.

Construction. Join OX. Draw OM perp. to XY then M is the middle pt. of XY.

Proof. Because the angles at A′ and M are right angles.

∴ the pts. M, O, A′, B are concyclic.

$$\therefore OA.AA' = MA.AB. \qquad [\text{IV. } 11$$

By def., $OX^2 = OA.OA'$; also $OX^2 = OM^2 + MX^2$;

$$\therefore OM^2 + MX^2 = OA.OA' = \begin{cases} OA(OA - AA') & \text{as in Fig. 1} \\ \text{or } OA(OA + AA') & \text{as in Fig. 2} \end{cases}$$

$$= OA^2 \mp OA.AA' = OM^2 + MA^2 \mp OA.AA'.$$

$$\therefore MX^2 = MA^2 \mp OA.AA' = MA^2 \mp MA.AB \qquad (\textit{Proved})$$
$$= MA(MA \mp AB) = MA.MB.$$

Hence the line is cut harmonically at

A, X, B, Y [*App.* V. (9)].

Q. E. D.

Cor. *If from any point* B *on the polar of a point* **A** *a pair of tangents be drawn to a circle, the chord of contact is divided harmonically by* A *and the polar of* A.

Let **XY** be the chord of contact.

Then **XY** is the polar of **B** [*App.* V. (52)];

∴ it passes through **A**,

∵ **B** is on the polar of **A** [*App.* V. (55)].

Hence the chord **XY** through **A** is cut harmonically by **A** and the polar of **A**. [*App.* V. (55)].

X. CONCURRENCE AND COLLINEARITY

56. THEOREM. *If* X, Y, Z *are points in the sides* BC, CA *and* AB *of a triangle* ABC, *such that the perpendiculars to the sides at* X, Y *and* Z *are concurrent, then*

$$(BX^2-XC^2)+(CY^2-YA^2)+(AZ^2-ZB^2)=0,$$

Conversely, *if*

$$(BX^2-XC^2)+(CY^2-YA^2)+(AZ^2-ZB^2)=0,$$

then the perpendiculars to the sides at X, Y *and* Z *are concurrent.*

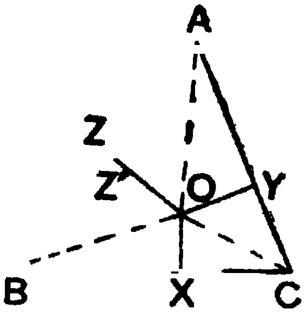


Fig. 1.

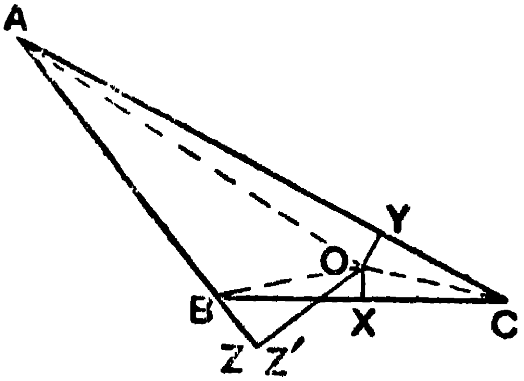


Fig. 2.

(i) Let the perps. meet at O. Join OA, OB, OC.

Now,

$$BX^2-XC^2=(OB^2-OX^2)-(OC^2-OX^2)=OB^2-OC^2;$$

Similarly,

$CY^2-YA^2=OC^2-OA^2$ and $AZ^2-ZB^2=OA^2-OB^2$.

Hence $(BX^2-XC^2)+(CY^2-YA^2)+(AZ^2-ZB^2)$

$$=(OB^2-OC^2)+(OC^2-OA^2)+(OA^2-OB^2)=0.$$

(ii) *Conversely*, let the perps. XO, YO meet at O; then OZ shall be perp. to AB. If not, let OZ′ be perp. AB.

Then $(BX^2-XC^2)+(CY^2-YA^2)+(AZ'^2-Z'B^2)=0.$

By *Hyp.*, $(BX^2-XC^2)+(CY^2-YA^2)+(AZ^2-ZB^2)=0.$

$$\therefore AZ'^2-Z'B^2=AZ^2-ZB^2;$$

i.e., $(AZ'+Z'B)(AZ'-Z'B)=(AZ+ZB)(AZ-ZB).$ (α)

Now in Fig. 1, $AZ' + Z'B = AB = AZ + ZB$;

$\therefore\ AZ' - Z'B = \qquad AZ - ZB$;

whence $2\,AZ' = 2\,AZ$.

And in Fig. 2, $AZ' - Z'B = AB = AZ - ZB$;

$\therefore$ from (α), $AZ' + Z'B = \qquad AZ + ZB$;

whence $2\,AZ' = 2\,AZ$.

Thus in both cases $AZ' = AZ$, *i.e.*, Z' coincides with **Z**.
Hence the perps. **XO**, **YO** and **ZO** are concurrent. Q. E. D.

Rider 1. The three altitudes of a triangle are concurrent.

Let **AP**, **BQ**, **CR** be the altitudes [see Fig. *App.* III, (7)].

By *App.* IV. (3), *Cor.*,

$$BP^2 - PC^2 = AB^2 - AC^2,\ CQ^2 \quad QA^2 = BC^2 - AB^2,$$
$$AR^2 - RB^2 = AC^2 - BC^2.$$

Hence $(BP^2 - PC^2) + (CQ^2 - QA^2) + (AR^2 - RB^2) = 0$ identically.

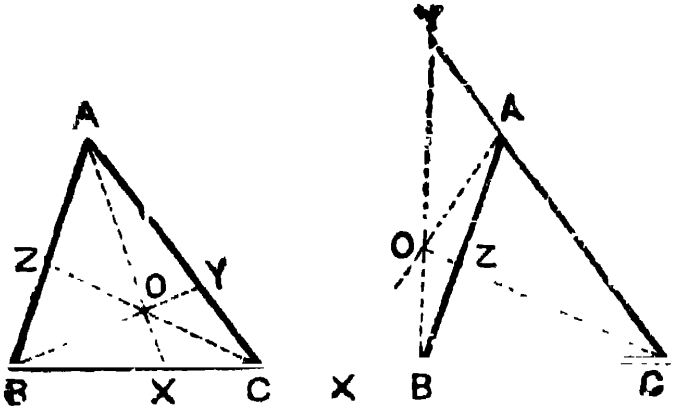
$\therefore$ **AP**, **BQ** and **CR** are concurrent.

Rider 2. The perpendiculars at the middle points of the sides of a triangle are concurrent.

Rider 3. The three common tangents at the points of contact of three circles that touch one another are concurrent.

Rider 4. The perpendiculars at the points of contact of each of the excircles are concurrent.

57. Ceva's Theorem. *If three straight lines drawn from the vertices of a triangle to meet the opposite sides are concurrent, then the product of the three ratios of the segments into which they divide the sides is equal to 1.*



Let the st. lines **AX**, **BY**, **CZ**, which meet the opposite sides **X**, **Y** and **Z**, be concurrent at **O**.

To prove that $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

Proof. Because the Δ^s AOB, AOC are on the same base OA and BC joins their vertices ;

$$\therefore \frac{BX}{XC} = \frac{\Delta\ AOB}{\Delta\ AOC}. \qquad [\textit{App.}\ IV.\ (2).$$

Similarly $\frac{CY}{YA} = \frac{\Delta\ BOC}{\Delta\ AOB}$ and $\frac{AZ}{ZB} = \frac{\Delta\ AOC}{\Delta\ BOC}$.

Multiply these ratios ; thus

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{\Delta\ AOB}{\Delta\ AOC} \cdot \frac{\Delta\ BOC}{\Delta\ AOB} \cdot \frac{\Delta\ AOC}{\Delta\ BOC} = 1.$$

Q. E. D.

Note. This result may be put in the form

$$BX.CY.AZ = XC.YA.ZB.$$

Hence Ceva's Theorem may be otherwise enunciated thus :—*If three straight lines drawn from the vertices of a triangle to meet the opposite sides are concurrent, then the product of three alternate segments is equal to the product of the three remaining segments.*

58. THEOREM. *If three straight lines, drawn from the vertices of a triangle, meet the opposite sides and make the product of the three ratios of the segments into which they divide the sides equal to 1, then the three straight lines are concurrent.* [Converse of (57)].

[Prove by *reductio ad absurdum.*]

59. **Menelaus' Theorem.** *If a transversal cuts the sides of a triangle, then the product of the three ratios of the segments into which it divides the sides is equal to 1.*

Let the transversal XYZ cut the sides BC, CA and AB of the Δ ABC in X, Y and Z.

To prove that $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$.

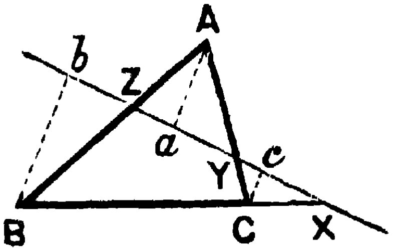


Fig. 1.

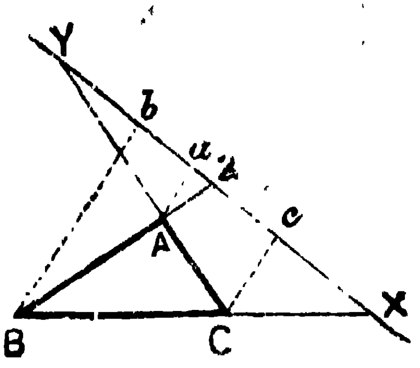


Fig. 2.

Draw A*a*, B*b*, C*c* perp. to XYZ, and denote them respectively by *l*, *m*, *n*.

Proof. Because Δ^s X*b*B, X*c*C are simlr.; ∴ $\frac{BX}{XC}=\frac{m}{n}$.

... Δ^s Y*c*C, Y*a*A. $\frac{CY}{YA}=\frac{n}{l}$.

... Δ^s Z*a*A, Z*b*B. $\frac{AZ}{ZB}=\frac{l}{m}$.

Multiplying these ratios together,

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=\frac{m}{n}\cdot\frac{n}{l}\cdot\frac{l}{m}=1,$$ Q. E. D.

Note. From this result, we have

$$BX.CY.AZ=XC.YA.ZB.$$

Thus Menelaus' Theorem may be otherwise enunciated thus :—*If a transversal cuts the sides of a triangle, then the product of three alternate segments is equal to the product of the three remaining segments.*

Obs. The student should notice that if a transversal cuts the sides of a triangle, it must cut either two of the sides *internally* and third side *externally* (Fig. 1), or all the three sides *externally* (Fig. 2).

60. THEOREM. *If three points are taken in the three sides of a triangle (one in each) and if the product of the three ratios of the segments into which they divide the sides is equal to 1, then the three points lie on a transversal, that is, they are collinear.* [Converse of (59.)]

[Prove by *reductio ad absurdum.*]

Rider. In a triangle, the internal bisectors of any two angles and the external bisector of the third angle meet the opposite sides in collinear points.

61. THEOREM. *If any three concurrent lines* AX, BY, CZ, *from the vertices of a triangle* ABC *meet the opposite sides, in* X, Y, Z, *and if* YZ *cut* BC *in* X′, *then* X, X′ *are harmonic conjugates of* B *and* C.

Proof. Because AX, BY, CZ are concurrent [Figs. of (57)];

$$\therefore \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1 \quad [\textit{Ceva.}]$$

Also ∵ YZ is a transversal, cutting BC in X′,

$$\therefore \frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1 \quad [\textit{Menelaus.}]$$

$$\therefore \frac{BX}{XC} = \frac{BX'}{X'C};$$

that is, B, X, C, X′ form a harmonic range. Q. E. D.

Note. Similarly it may be proved that if ZX cut AC in Y′, and XY cut AB in Z′, then Y, Y′ are harmonic conjugates of A and C, and Z, Z′ of A and B.

Ex. 46. In a triangle, prove by Ceva's Theorem, that (i) the medians, (ii) the bisectors of angles, (iii) the altitudes, meet in a point.

Ex. 47. The joins of the points of contact of the in-circle to the opposite vertices are concurrent.

Ex. 48. The joins of the points of contact of each of the ex-circles to the opposite vertices are concurrent.

Ex. 49. In a triangle, the internal bisector of an angle and the external bisectors of the other two angles are concurrent.

XI. THE COMPLETE QUADRILATERAL

DEFINITIONS

If pairs of opposite sides of a quadrilateral are produced to meet, the figure thus formed is called a **complete quadrilateral** or **a tetragram.** [See Fig. of (62).]

Thus if AB, DC meet at E, and AD, BC at F, then the figure AECF is called a complete quadrilateral.

The intersection of any two sides is called a **vertex** ; thus A, B, C, D, E and F are vertices. **Opposite vertices** are those which do not lie on the same side ; thus A, C ; B, D ; and E, F are opposite vertices.

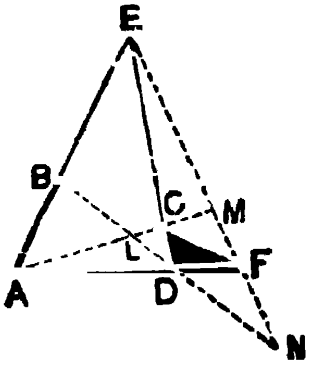
The join of any two opposite vertices is called a **diagonal** ; thus AC, BD and EF are the diagonals, EF being called the **third diagonal.**

The triangle formed by the three diagonals is called the **diagonal triangle** ; thus LMN is the diagonal triangle.

Hence a complete quadrilateral has *six* vertices and *three* diagonals, and consists of *seven* lines that intersect in *nine* points.

62. THEOREM. *In a complete quadrilateral, each diagonal is divided harmonically by the other two.*

Let AECF be a complete quadrilateral.



To prove that EF is cut harmonically by AC, BD at M and N.

Proof. In the $\triangle$ **AEF**, **AM**, **ED** and **FB** are concurrent;

$$\therefore \frac{AB}{BE} \cdot \frac{EM}{MF} \cdot \frac{FD}{DA} = 1 \text{ [*Ceva.*]}$$

Again, the transversal **BDN** cuts the sides of the $\triangle$ **AEF** at **B**, **D** and **N**;

$$\therefore \frac{AB}{BE} \cdot \frac{EN}{NF} \cdot \frac{FD}{DA} = 1 \text{ [*Menelaus*]}$$

$$\text{Hence } \frac{EM}{MF} = \frac{EN}{NF};$$

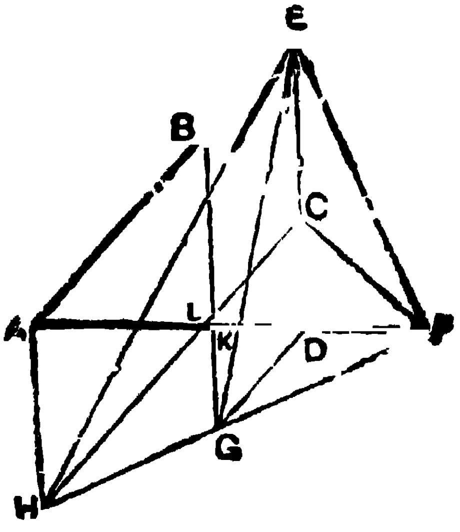
that is, **EF** is divided harmonically at **M** and **N**.

Similarly by considering the $\triangle$s **AEC**, **BCD**, it can be proved that **AC** and **BD** are divided harmonically at **L**, **M** and **L**, **N** respectively. [Prove this.] Q. E. D.

Ex 50. In the figure of the Theorem, if **EL** cut **BC** in **P** and **AD** in **Q**, prove that every range of four points is a harmonic range.

63. THEOREM. *In a complete quadrilateral, the middle points of three diagonals are collinear.*

Let **AECF** be a complete quadrilateral.



To prove that the middle points of **AC**, **BD** and **EF** are collinear.

Complete the parms. **EBGD**, and **ECHA**.

Let BG, CH cut AD at K, L.

Proof. Since CD is ∥ to BK and LC to AB;

$$\therefore \frac{FD}{DK} = \frac{FC}{CB} = \frac{FL}{LA}; \text{ hence } \frac{FD}{DK} = \frac{FL}{LA};$$

$$\therefore \text{ alternately, } \frac{FD}{FL} = \frac{DK}{LA} = \frac{DG}{LH}, \text{ by similar } \triangle^s \text{ DKG, LAH.}$$

$$\therefore \frac{FD}{FL} = \frac{DG}{LH}, \text{ and DG is } \parallel \text{ to LH};$$

∴ FGH is one st. line [(LXXVI), 6].

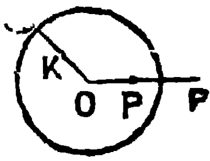
Hence the mid pts. of EF, EG and EH lie on a parallel to FGH.

But the mid pts. of EH and EG are also the mid pts. of AC and BD, ∵ EH, AC and EG, BD are diagonals of parms.;

∴ mid pts. of AC, BD and EF are collinear. Q. E. D.

XII. INVERSION

Def. If through a fixed point, a straight line OP is drawn and a point P′ be taken on OP, (produced, if necessary), such that OP.OP′, $=k^2$, where k is constant, then of the points P, P′, each is said to be the **inverse** of the other with respect to the circle whose centre is O and radius k.



The circle is called the **circle of inversion**, its centre O is called the **centre of inversion**, and radius k the **radius of inversion**. Sometimes k^2 is called the **constant of inversion**.

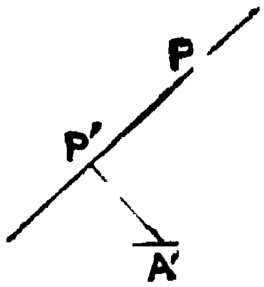
If P moves and describes a locus, then to every position at P on the locus, there shall be a *corresponding* position of its *inverse* P′. Hence P′ will describe another locus, which is called the *inverse* of the locus of P, with respect to the circle of inversion. Thus if P moves on a straight line, its inverse P′ may move on a circle [See *App.* V. (55)].

In this case therefore the circle is said to be the inverse of the straight line.

64. THEOREM. *The inverse of a straight line passing through the centre of inversion is the straight line itself.*

For if P moves along a straight line through O, P must move alone that line.

65. THEOREM. *The inverse of a straight line not through the centre of inversion is a circle through the centre of inversion.*



Let P be any pt. in PA, O the centre and k the radius of inversion.

Draw OA perp. to PA. Let P′, A′ be the inverses of P and A. Join P′A′.

Proof Then OA is a fixed line and OA. OA′ $= k^2$; $\therefore$ OA′ is of fixed length.
Also OA.OA′ $= k^2 =$ OP.OP′ ;

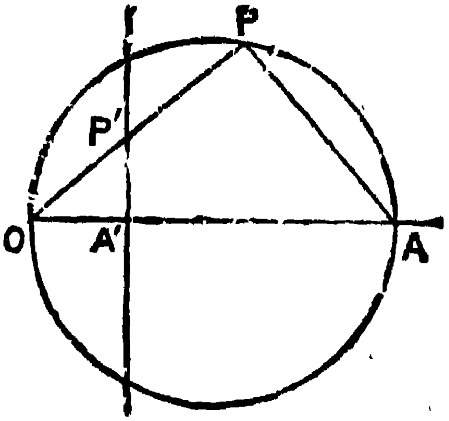
$\therefore$ A, A′, P′, P are concyclic ;

$\therefore$ $\angle$OP′A′ $=$ $\angle$OAP $=$ a right angle.

Hence the locus of P′ is the circle OP′A′, which passes through O and has the diameter OA′ perpendicular to PA.

Q. E. D.

66. THEOREM. *The inverse of a circle passing through the centre of inversion is a straight line perpendicular to the diameter through the centre of inversion.*



Let O be the centre and k the radius of inversion ; and OA the diameter of the given circle through O. Let P be any pt. on the circle ; and A′, P′ be the inverses of A and P.

Proof. Then OA is a fixed length and OA.OA′ $= k^2$;

$\therefore$ A′ is a fixed pt.

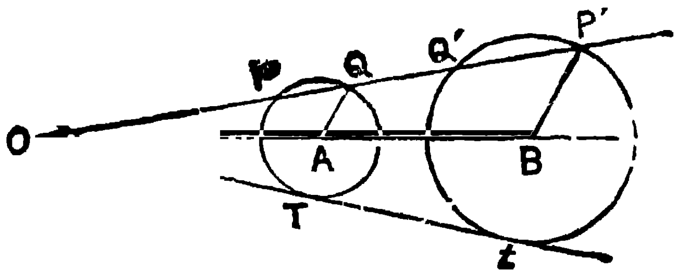
Also $OA.OA' = k^2 = OP.OP'$;

$\therefore$ A, A', P', P are concyclic ;

$\therefore$ $\angle OA'P' = \angle OPA$ = a right angle.

Hence the locus of P' is the fixed line P'A' perpendicular to the diamater through O. Q. E. D.

67. THEOREM. *The inverse of a circle not through the centre of inversion is a circle.*



Let (A) be the given circle and P any pt. on it ; and let O be the centre and k the radius of inversion.

Draw the tangent OT to (A).

Let OP cut the given circle again at Q, and let P', Q' be the inverses of P and Q. Join AQ, and draw P'B parallel to QA to meet OA (produced, if necessary) at B.

Proof. Then $OP.OP' = k^2$, and $OP.OQ = OT^2$.

$$\therefore \frac{OP.OP'}{OP.OQ} = \frac{k^2}{OT^2}\text{; that is, } \frac{OP'}{OQ} = \text{constant.}$$

Now the Δ^s OP'B, OQA are similar ;

$\therefore$ $\begin{cases} OB : OA = OP' : OQ = \text{const. ; } i.e., \text{ B is a fixed pt. ;} \\ BP' : AQ = OP' : OQ = \text{const. ; } i.e., \text{ BP' is of fixed length.} \end{cases}$

Hence the locus of P' is a circle with centre B and radius BP'. Q. E. D.

Cor. *The centre of inversion is a centre of similitude of a circle and its inverse.*

ANSWERS

IV. [p 20.]

1. (i) 72°, (ii) 60°, (iii) 45°, (iv) 36°, (v) 15°, (vi) 49° 48′, (vii) 33° 59′ 25″, (viii) 6° 35′ 1″, (ix) 89° 1′ 14″, (x) 89° 59′ 9″.

2. (i) 155°, (ii) 120°, (iii) 95°, (iv) 72°, (v) 60°, (vi) 30°, (vii) 59° 54′ 30″, (viii) 179° 41′ 10″, (ix) 179° 59′ 35″.

3 72°, 144°, 36°. **4** 120°, 60°. **5.** 67° 30′, 22° 30′.

7. 36, 144°.

XII. [p. 42]

1. 3·4 cm. **2.** 1·5 cm.

XIII. [p 51.]

1. ∠AGC = ∠DGH = ∠GHE = ∠BHF = 46°; ∠AGD = ∠CGH = ∠GHF = ∠BHE = 134°.

2. ∠AGD = ∠CGH = ∠GHF = ∠BHE = 144°; ∠AGC = ∠DGH = ∠GHE = ∠FHB = 36°.

3. 68°, 112°, 68°. **4** 3°. **5.** 70°, 60°, 50°.

6. 43°, 51°, 86°.

XIV. [p. 55]

1. 61°. **2** 55°. **3** 60°. **4** 87°, 75°. **5.** 36°, 72°, 72°.

6. 67°, 67°, 46°. **8.** 30°, 60°.

XV. [p. 61.]

1. 140°. **2.** 80°, 45°, 55°. **3** 48°, 132° 132°. **4.** 121°, 22°.

5. (i) 12 rt. ∠s; (ii) 16 rt. ∠s; (iii) 26 rt. ∠s. **6.** (i) 108°, 72°. (ii) 120°, 60°. **7.** 5.

8. No. **9** 6. **10.** 12. **11.** 8. **12.** 6.

13 7. **14.** 30°.

XVIII. [p. 72.]

20. 6. **21.** 50°, 75°, 110°, 150°, 160°, 175°. **22.** 6, 12.

XXII. [p. 84.]

13. 3 cm. **14.** ·47 cm, ·94 cm., 1·41 cm., 1·88 cm., 2·35 cm. 2·82 cm., 3·29 cm., 3·76 cm., 4·23 cm.

15. 250 yds., 400 yds., 45 yds.

XXIX. [p. 101.]

3. 2·4 cm., 8·4 cm. 4. 4·2 cm., 7 cm.

XXX. [p 108]

1. a = 3·12 cm., they are equal 2. 2 : 1. 3 BO = 2·7 cm. ; AO = 3·1 cm, ; ·86. The values are equal. 4. 6 cm., 3·4 cm.
5. 10·5 cm. 6. AD = 2·57 cm., BD = 1 18 cm. OD = 1·74 cm. ; the two results are very nearly equal. 7. Base = 14 cm.

XXXI. [p. 115.]

2. 8 cm. 3. Each diagonal = 6·65 cm., each angle = 90°.
5. 1·6″, 1·9″. 6. 3·54 cm. 7. 1·6″. 8. 2·8 cm. 9. 6·8 cm.

XXXII. [p. 141.]

1. (4, 0) ; (−3, 0) ; (0, 4) ; (0, −5) ; (6, 4) ; (−5, 3) ; (−6, −7) ; (6, −6) ; (0, 0).
4. x-axis. 5. y-axis. 6. Parallel to y-axis.
7. Parallel to x-axis. 8. Parl. to y-axis. 9. Parl. to x-axis.
10. 1·8 sq. in. 11. ·56 sq. in. 12. ·66 sq. in.
13. 1·04 sq. in. 14. ·88 sq. in.

XXXIII. [p. 145.]

1. (i) 161·28 sq. in., 1·12 sq. ft. ; (ii) 180 sq. mm. 1·8 sq. cm. (iii) 69·75 sq. cm, 6975 sq. mm. ; (iv) $9\frac{1}{6}$, 1320 ; (v) 7·35075, 73507500000 ; (vi) mn, $\frac{mn}{1296}$; (vii) 6 ab.
2. (i) $12\frac{1}{4}$. (ii) 29·16 ; (iii) 576 ; (iv) $\frac{9}{16}$; (v) $9x^2$.
3. (i) 6 ft. ; (ii) 2 cm. ; (iii) 5·5 yds. ; (iv) $4a$ ft.
4. (i) 17·28, 1728 ; (ii) $11\frac{1}{3}$, 1632 ; (iii) 6090, 60·9.
5. 175 yds. 9. 91000 sq. yds. 10. 1 in. to 10 in. 11. 65, 130.
12. 220. 13. A to B Rs. 250. 14. (i) 20 ; (ii) 40 ; (iii) 28.

XXXIV. [p. 150.]

9. (i) 4·87 sq. in. ; (ii) 10·85 sq. cm. ; (iii) 10·22 sq. cm ; (iv) 20 sq. cm. 10. 13 8 sq. cm.

XXXV. [p. 154.]

1. (i) 9·9 sq. cm. ; (ii) 1·9 sq. cm. ; (iii) 24 sq. cm. ; (iv) ·6 sq. in.
2. (i) 52 sq. units ; (ii) 130 sq. units. ; (iii) 16·5 sq. units ; (iv) 35 sq. units.

XXXVIII. [p. 164.]

1. (i) 15 ; (ii) 8·5 ; (iii) 10·63 ; (iv) 1·3 ; (v) 19·09 ; (vi) 23·86.

2. (i) 8, 60 sq. units ; (ii) 48, 336 sq. units ; (iii) 10·84, 66·68 sq. units ; (iv) 14·3, 17·16 sq. units. **3.** (i) ·5 in. ; (ii) 1·4 in. ; (iii) 1·7 in. ; (iv) 1 in. ; (v) 2 in. ; (vi) 1·64 in. ; (vii) 2·32 in. ; (viii) 1·71 in.

4. ·95 in. ; ·76 in. ; ·63 in.

5. 23·04 cm. ; 9 cm. **6.** (i) 79·69 ; (ii) 12.

7. (i) 49·50 ; (ii) 76·65. **8.** (i) 10·63 ; 112·99 sq. units ; (ii) 31·91 ; 1018·24 sq. units. **9.** 10·39 cm. ; 62·24 sq. cm.

10. (i) 19·5 ; (ii) 15. **11.** 33·54 ft. **12.** 156 yds.

13. 37 yds. **14** 30 ft. **15.** 72 ft. **16.** 44 ft **17.** 25 ft.

XLIII. [p. 200.]

1. 2·3 cm. **2.** 1·6″. **3** 8·5 cm. **4.** 4·5 cm. ; 8·9 cm.

5. 13″. **6.** 12″.

XLIV. [p. 204.]

1. 2·7 cm. **2.** 1·5″. **3.** 8·14″; yes.

XLV. [p. 207.]

1. 18·3″ ; 8″. **6.** 4 cm. nearly. **7.** 7 cm. or 23 cm.

8. 48 cm., 7 cm.

XLVII. [p. 214.]

1 150°. **2.** 105°.

XLIX. [p. 220]

(i) [illegible] and (ii) [illegible] of the circumference.

LIV. [p. 236]

2. 8 cm. **3.** (i) 5 cm. ; (ii) 5 cm.

LV. [p. 238.]

1. 5 cm. **2.** (i) 2″ ; (ii) 2·4″; (iii) 1·5″.

3. (i) 13·4 cm. ; (ii) 12 cm. **4.** 13·4 cm.

LVI. [p. 241.]

2. (i) 2 cm. ; (ii) 10 cm. **3.** 7, 8 and 9 cm.

4. 4·5, ·7 and 1·6 cm.

LVII. [p. 245.]

1. 110° ; 25. **2.** 102° ; 78°. **3.** 70°.

LIX. [p. 249.]

1. 2″. **2.** 3·2 cm. **3.** 483 ft. **4.** 4·84 cm ; 4·3 cm.

LX. [p. 252.]

4. 3·9 cm. ; 1·9 cm. **5.** 8·01 ft.

LXII [p. 262.]

5. 20·7 sq. cm. ; 82·8 sq. cm. **6.** 9″ ; 1·3″.

LXIV. [p. 308.]

1. 9·22 cm. **2.** 10·63 cm. **3.** (i) 16·73 cm. ; (ii) 11·66 cm.
4. 15·65 cm. **5** 4·5 cm. ; ·63 cm. ; 2·25 cm. **6.** 5·5 in.
7. 1·3 in. ; 1·25 in. **8.** 3·8 in. ; 1·52 in. **10.** 14·14 sq. ft.
11. 23·4 sq. cm. **12.** 68·7 sq. in.

LXV. [p. 312.]

1. 5·6 cm. **2.** 1·87 cm. ; 4·31 cm ; 4·68 cm. **3.** 5·56 cm.

LXVI. [p. 321.]

1. 7·4 cm. **2.** 8″ nearly **3** 1·1″. **4.** 5·3 cm.
5. ·4″. **8.** 3 cm. or 8 cm

LXVIII. [p. 329.]

4. 3·09 cm ; 1·91 cm. **5.** 8·09 cm. ; 13·09 cm.

LXIX. [p. 332.]

1. 6 cm. **2.** 3·22″. **3** 45″. **4** 3·5 cm.

LXIX (a) [p. 334.]

1. 6·6, 1·4. **2.** 4·9, 1·9. **3.** 12, 3. **4.** 8, 2.
5. 6·6, 2·4. **6** 4·85, 1·85. **7.** 9 or 4. **8.** 5·2 or ·8.
9. 6 or 6. **10.** 14·2 or 1·8. **11.** 8 or −2. **12.** 6·4 or −1·4.
13. 5 or 2. **14.** 8·7 or ·3.

MATRICULATION EXAMINATION PAPERS
CALCUTTA UNIVERSITY

Compulsory Papers

1927

1. Book I, Theorem 11.

Prove that the sum of the distances of any point from the three angular points of a triangle is greater than half of its perimeter. [*Page* 39, Ex. 5.

Or,

Book I, Theorem 24.

One angle of a parallelogram is a right angle Prove that it is a rectangle. [*Page* 52, Ex. 9.

2. Book III, Theorem 8 (i).

A circle is described on the hypotenuse of a right-angled triangle as diameter. Prove that the circle passes through the opposite angular point. [*Page* 220, Ex. 3.

Book III, Problem 1.

1928

1 Book I, Theorem 10.

Hence deduce that the hypotenuse is the greatest side of a right-angled triangle. [*Page* 37, Ex. 1.

Or,

Book I, Theorem 16.

If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled. [I 16, *Cor.* 3.

2. Book III, Theorem 12 A

Two equal circles intersect at **A** and **B** ; and through **A** any st. line **PAQ** is drawn terminated by the circumferences. Show that **BP=BQ**. [*Page* 230, Ex. 2

3. Book III, Theorem 5.

1929

1. Book I, Theorem 3.

Two st. lines **AB** and **CD** intersect at **E**. If the bisector of the angle **AEC** be produced, prove that it will bisect the ∠ **BED**. [*Page* 19, Ex. 1.

Or,

Book I, Theorem 19.

The triangle **ABC** has the angles at **B** and **C** equal. Show that the bisectors of these equal angles terminated by the opposite sides are equal.

2. Book III, Theorem 15.

If the circumference of a circle is divided into three equal arcs, the tangents drawn to the circle at the point of section form an equilateral triangle.

3. Book III, Problem 2.

[Traces of construction must be given, but no justification is required.]

1930

1. Book I, Theorem 16.

Find in degrees each angle of a polygon of five sides. Give reasons for your answer.*

Or,

Book II, Theorem 3, *Cor.* 2.

ABCD is any parallelogram and **O** is any point within it. Shew that the sum of the areas of the triangles **AOB** and **COD** is equal to half the area of the parallelogram. [*Page* 156, Ex. 7.

2. Book IV, Theorem 7.

In a triangle **ABC**, **AD** is perpendicular drawn to the base **BC**, and **O** is the middle point of **BC**.

Prove that the difference $AB^2 \sim AC^2 = 2BC.OD$. [*App.* IV, (3).

Or,

Book III, Theorem 13

The radius of a given circle is 1·5 in. Prove that all points from which the tangents drawn to the circle are of constant length 2 in. lie on a circle. Draw a diagram as accurately as you can. [*Page* 236, Ex. 11.

3. The plan of an orchard can be drawn by joining in order the points **A** (3, 4), **B** (5, −1), **C** (−2, −4) and **D** (−6, 2). Draw it on squared paper, taking five small divisions of your paper as unit. An artificial fountain is known to be situated at the intersection of **AC** and **BD**. Find the co-ordinates of the position of the fountain.

Construct a triangle whose base will be 6 cm. and the other two sides 3 and 5 cm. respectively. Measure as accurately as possible the altitude of the triangle.

[Traces and statement of construction are required.]

1931

1. Book I, Theorem 19.

A diagonal of a parallelogram is bisected, and through the point of bisection a straight line is drawn to be terminated by one pair of opposite sides. Show that the straight line is bisected at the point. [*See page* 64, Ex. 9 (ii).

Or,

Book I, Theorem 11.

Prove that the difference of any two sides of a triangle is less than the third side. [*Page* 39, Ex. 1.

2. Book IV, Theorem 4.

Prove that the square on a straight line is equal to four times the square on half the line. [*Page* 190, Ex. 4.

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* *Indeterminate.*

Or,

Book III, Problem 2.

A quadrilateral is described touching a circle. Prove that the sum of any pair of opposite sides is equal to the sum of the other pair. [*Page* 239, Ex. 10.

3. Construct a triangle, given the base, one side and the area.

1932

1. Book I, Theorem 8.

Show that it is impossible to draw three equal straight lines from a given point to a given straight line.

Or,

Book I, Theorem 14 (ii).

Prove that, if the three sides of one triangle are parallel to the three sides of another triangle, the corresponding angles are equal.

2. Book III, Theorem 1 (i).

Show how to construct a circle of given radius to pass through two given points. When is this construction impossible ? [*Page* 201, Ex. 21.

Or,

Book III, Theorem 13.

Show how to draw a tangent to a given circle parallel to a given st. line. How many such tangents are possible ? [*Page* 249, Ex. 8 (i).

3. Book I, Problem 17.

Divide the area of a given square into parts from which two equal squares can be made up.

1933

1. Book II, Theorem 5.

Prove that in an equilateral triangle four times the square on the perpendicular drawn from the vertex on the opposite side is equal to three times the square on any side. [*Page* 165, Ex. 28

Or,

Book IV, Theorem 8.

Prove that a triangle whose sides are 2, 3 and 4 inches is an obtuse-angled triangle. [*See page* 309, Ex. 6.

2. Book III, Theorem 3.

Find the locus of the mid-points of chords of constant length in a circle. [*Page* 236, Ex. 7.

Or,

Book III, Theorem 2.

Prove that two different circles cannot cut each other at more than two points. [III. 2, *Cor.* 2.

3. Book II, Problem 1.

[Traces only are required.]

Construct a rhombus equal in area to a given rectangle and having a side equal to a side of the rectangle.

[Traces only are required.]

1934

1. Book I, Theorem 9.

Show that the difference of any two sides of a triangle is less than the third side. [*Page* 39, Ex. 1.

Or,

Book II, Theorem 3, *Cor.* 1.

Show that the straight line joining the middle points of two sides of a triangle is parallel to the third side. [*App.* I. (2)

2. Book III, Theorem 5.

L is any point on the arc **PM** of a circle. The angles **LPM** and **LMP** are bisected by straight lines which intersect at **O.** Find the locus of the point **O.**

2. Book II, Problem 2.

Bisect a quadrilateral by a straight line drawn through an angular point. [*Page* 327, Ex. 3.

Or,

Book I, Problem 14.
(Traces only are required.)
Book IV, Problem 1.
Traces only are required.)

1935

1. Book I, Theorem 7.

Show that the diagonals of a rhombus bisect one another at rt. angles. [*Page* 31, Ex. 6.

Or,

Book III, Theorem 3.

Through a given pt. within a circle draw the least possible chord. [*Page* 211, Ex. 3.

2. Book IV, Theorem 8.

Book IV, Theorem 10.

Or,

Book IV, Theorem 11.

ABC is a **Δ** rt. ∠d at **C** ; from **C** a perp. **CD** is drawn to the hypotenuse ; show that the sq. on **CD**=rect. **AD.DB**. [*Page* 356. Ex. 6.

3. Book II, Problem 1.

Describe a rhombus equal to a given parm. and standing on the same base. When does the construction fail ?

Additional Papers

1927

1. Describe a circle so as to touch externally a given circle and a st. line at a given point. [*Page* 349, Ex. 12.

Or,

Appendix III. (12).

2. Book V, Theorem 14 (i).

D is the middle point of **BC**, the base of a triangle **ABC**; **DE**, **DF** bisect the angle **ADB**, **ADC** meeting **AB**, **AC** respectively in **E**, **F**. Show that **EF** is parallel to **BC**. [*Page* 401, Ex. 4.

Or,

Book V, Theorem 2 (i).

OPQ is a st. line drawn through a fixed point **O** and is such that **OP** : **OQ** is constant. If **P** moves along a fixed st. line, find the locus of **Q**. [*Page* 373, Ex. 8.

3. *Appendix* III. (11).

1928

1. Construct a circle of given radius so as to touch two given intersecting st. lines. [*Page* 346, Ex. 13.

[Only traces of construction are required.]

2. Book V, Theorem 10.

ABC is a triangle inscribed in a circle. If the tangent at **A** meets **BC** produced in **D**, prove that $CD : BD = AC^2 : AB^2$.

[*Page* 391, Ex. 12.

Or,

Book V, Theorem 15 (i).

D is the middle point of **BC** of a triangle **ABC**; **DE**, **DF** bisect the angle **ADB**, **ADC** meeting **AC** and **AB** produced in **E** and **F**; prove that **EF** is parallel to **CB**. [*Page* 401, Ex. 4.

3. Find a point within a triangle so that the angles subtended at it by the three sides may be equal.

1929

1. Book V, Theorem 2.

Show how to divide a given st. line into three equal parts.

[*Page* 101, Ex. 1

Or,

Book V, Theorem 3, *Cor.*

Prove that a common tangent to two circles divides the line joining their centres internally or externally in the ratio of their radii.

2. Inscribe a regular hexagon in a given circle. [*Page* 263, Ex. 9. (iii)

[Traces of construction are required.]

3. Book III, Theorem 6.

Appendix III, (12).

1930

1. Book V, Theorem 14.

O is the in-centre of the triangle **ABC.** Measure the length of '**OA**, if **AB** = 2 in. **BC** = 3 in., and **AC** = 4 in.

Or,

Book V, Theorem 11.

Two equilateral triangles are described in two given circles; prove that the ratio of the areas of the triangles is the same as that of the squares on the radii. [*Page* 391, Ex. 7 (iii).

9. Book IV, Problem 5.

Or,

Book III, Problem 6.

[Traces and statement of construction are required]

3. Book IV, Theorem 15.

If **O** be the orthocentre of the triangle **ABC,** prove that the circle round any of the triangles **OAB, OBC, OCA** is equal to that round the triangle **ABC.** [*Page* 272, Ex. 6.

1931

1. Book IV, Theorem 10.

Prove that the sum of the squares on the four sides of a parm. is equal to the sum of the squares on the diagonals. [*Page* 313, Ex. 10.

Or,

Book III, Problem 3.

How many common tangents may be drawn when the circles cut one another and how many to two non-intersecting circles? When is no common tangent possible? [*Page* 252, Ex. 3.

2. Book V, Theorem 3, *Cor.*

ABC is a triangle right-angled at **A**, and **AD** is drawn perp. to **BC.** If **AB** = 5 cm. and **AC** = 12 cm., find **BD** and **CD.** [*See* V, 8, *Cor.*

Or,

Book V, Theorem 10.

Equilateral triangles are described on the sides of a right-angled triangle. Prove that the area of the triangle on the hypotenuse is equal to the sum of the areas of the other two triangles. [*Page* 430, Ex. 47.

3. Prove that the locus of a pt. from which tangents drawn to two given circles are equal is a st. line. [*App.* V. (22).

If **A** and **B** are the centres of the circles, and **P** the point where the above locus intersects **AB**, find **PA** and **PB**, being given that **AB** = 10 cm. and the radii of the circles 5 cm. and 3 cm. respectively.

1932

1. Book V, Theorem 14 (i).

The base of a triangle is 3 in., the ratio of the sides is 1 : 3, and the vertical angle is 60°. Construct the triangle.

Or,

Book V, Theorem 10

Bisect a given △ by a st. line drawn parl. to the base. [*Page* 421, (4).

2. Book V, Theorem 3.

From a pt. on a river-bank a light is observed st. across on the other bank and at the level of the eye. Give a simple geometrical construction, on the basis of the above proposition, by which the distance of the light may be ascertained without crossing the river.

3. Inscribe a regular hexagon in a given circle. [*Page* 263, Ex. 9 (iii).

Prove that if the alternate vertices are joined, the area of the triangle thus formed is half of that of the hexagon.

Or,

Prove that the circumference of a circle is greater than three times the diameter.

1933

2. Book IV, Problem 7.

Construction an angle equal to a fifth part of a right angle.

(Traces only are required.)

Or,

Appendix V, (35).

1. Book III, Theorem 1 (ii).

A goat is tied by a length of rope to a peg on the ground so that the goat can almost reach a distance l from the peg. If the peg is at a distance d ($< l$) from a st. row of plants, show that the goat can feed of a length of the row equal to $2\sqrt{l^2-d^2}$.

3. Book IV, Theorem 10.

Show that three times the sum of the sqs. on the sides of a △ = four times the sum of the sqs. on the median. [*Page* 314, Ex. 19.

Or,

Book IV, Theorem 7.

Show that the rect. contained by the two segments into which a given st. line may be divided cannot exceed the sq. on half the line.

[*See* IV. 7, *Cor*. 1.

1934

Book IV, Theorem 11.

The tangents to two intersecting circles from any pt. on their common chord produced are equal. [*Page* 321, Ex. 10.

Or,

Prove that the orthocentre, the circumcentre, the centriod and the centre of the nine-point circle of a triangle are collinear.

[*App*. III, (18) Note.

2. Book V, Theorem 2 (ii).

Find the locus of the centriods of a △ of a constant area standing on the same base.

3. Draw a circle of diameter 3 in. and inscribe accurately a regular pentagon in it. Measure length of a side of the pentagon. [IV *Prob*. 8.

Or,

Appendix V, (34).

1935

1. Book III, Theorem 4.

If two equal chords intersect, prove that the segments of one are respectively equal to the segments of the other. [*Page* 208, [Ex. 12.

Or,

Draw a circle of radius 5 cm. Inscribe by geometrical methods a regular octagon in the circle. [III. Prob. 10 *Note*.

Show how to construct a circle to touch each of two parallel st. lines and a transversal. [*Page* 346, Ex. 17.

2. **P** is a pt. on an arc of a circle whose chord is **AB**. **AP** is produced to **Q**, so that **PQ**=**PB**. Find the locus of the middle point of **BQ**.

3. Book V, Theorem 10.

Two equiangular triangles have areas in the ratio 3 : 2 and an altitude of the greater is 5·2 cm. What is the corresponding altitude of the other.

Or,

P is a pt. lying between two intersecting st. lines **OA**, and **OB**. Triangles are formed with adjacent sides along **OA** and **OB** by drawing lines through **P**. Show that of these triangles the one of which this interior side is bisected at **P** has the minimum area. [*App*. V. (45).